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FOUR-NODED TRIANGULAR FINITE ELEMENTS

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ABSTRACT

The limits and extents of the isoparametric formulation for a four-noded finite element are derived by transforming the parametrized shape function from $\xi-\eta$ to $x-y$ coordinates. This analytic inversion results in a quadratic equation. The coefficients of this equation dictate whether the form of the shape function is a polynomial, a rational polynomial or an expression containing a square root. Furthermore, in $x-y$ coordinates the C^0 shape function for a triangular element with a mid-side node can be derived. Consequently, this study extends the class of elements for which closed form shape functions can be constructed. In 1975 E. L. Wachspress introduced a rational polynomial formulation for finite elements which applies consistently to any convex n-sided polygon; it does not apply to elements with a mid-side node. Combining these results, convergent shape functions and consistent strain matrices can be formulated for all non-concave quadrilaterals. The associated energy density function can be integrated exactly using the divergence theorem.

Keywords: Shape function, isoparametric, degenerate element, side-node.

FORM OF AN ELEMENT SHAPE FUNCTION

Rectangular and triangular C^0 shape functions can be represented as a polynomial in normed coordinates $\{\xi,\eta\}$.

$$\begin{array}{rll} \text{Triangle}: & \phi^t(\xi,\eta) = & a_3\eta + a_2\xi + a_1 \\ \text{Rectangle}: & \phi^r(\xi,\eta) = & b_4\eta\xi + b_3\eta + b_2\xi + b_1 \end{array} \tag{1}$$

The constant coefficients $\{a_1,a_2,a_3\}$ and $\{b_1,b_2,b_3,b_4\}$ are defined by the geometry of the domain and the general shape function requirements. Given a set of shape functions $\phi_i(x,y)$ evaluated at the vertex coordinate $\{x_j,y_j\}$; define the shape function such that each is single valued at its named coordinate:

$$\phi_i(x_j,y_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} . \tag{2}$$

Along the boundary adjacent to the single valued node the behavior should be linear. Let the adjacent node, traveling cyclically around the boundary, be node $\{x_k,y_k\}$:

$$\phi_k(x_jt + x_k(1-t), y_jt + y_k(1-t)) = t \tag{3}$$

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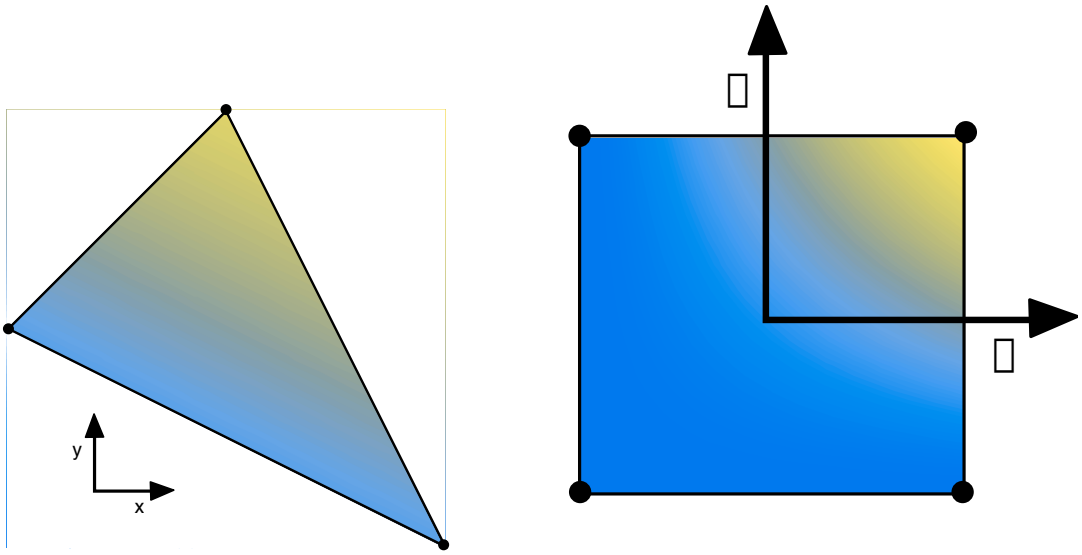


FIG. 1. Polynomial shape functions

Within the domain defined by the vertices $\mathcal{D}$, the shape function should be smooth, bounded and be able, in linear combination, to reproduce constant and linear fields, then $\forall\{x, y\} \in \mathcal{D}$ (Irons and Ahmad 1980):

$$0 < \phi_i(x, y) < 1, \quad \sum_{i=1}^{n} \phi_i(x, y) = 1, \quad \sum_{i=1}^{n} x_i \phi_i(x, y) = x \quad \text{and} \quad \sum_{i=1}^{n} y_i \phi_i(x, y) = y. \quad (4)$$

The shape functions for the Courant's triangle (1943) and Taig's quadrilateral (1961) are polynomial shape functions which satisfy these requirements, see figure 1:

$$\phi_a^t(\xi, \eta) = \frac{\triangle(\vec{\xi}, b, c)}{\triangle(a, b, c)}, \ \forall\{\xi, \eta\} \in \mathcal{D} \quad \text{and} \quad \phi_i^r(\xi, \eta) = \frac{(1 \pm \xi)(1 \pm \eta)}{4}, \ \forall\{\xi, \eta\} \in (-1, 1)$$

PERSPECTIVE TRANSFORMATION

Any shape function representation which is derived from a perspective transformation of a valid shape function also satisfies the general shape function requirements. A perspective transformation preserves the linearity of lines and the bounded nature of a solution (Rogers and Adams 1990). The triangular shape function formulation applies automatically to any planar, continuous domain defined by three nodes. The quadrilateral shape function must be transformed to apply to domain which is not square.

Given set of scaled shape function for a unit square:

$$N_1(s, t) = (1 - s)(1 - t), \quad N_2(s, t) = s(1 - t), \quad N_3(s, t) = st \quad \text{and} \quad N_4(s, t) = (1 - s)t \quad (5)$$

as a test case, transform shape function $N_3(s, t)$ using a perspective transformation:

$$\left\langle x^*, y^*, \hat{N}_3^*(x, y), h \right\rangle = \langle s, t, st, 1 \rangle \, [\mathbf{A}] \quad (6)$$

$$x = \frac{x^*}{h}, \quad y = \frac{y^*}{h} \quad \text{and} \quad \hat{N}_3(x, y) = \frac{\hat{N}_3^*(x, y)}{h}. \quad (7)$$

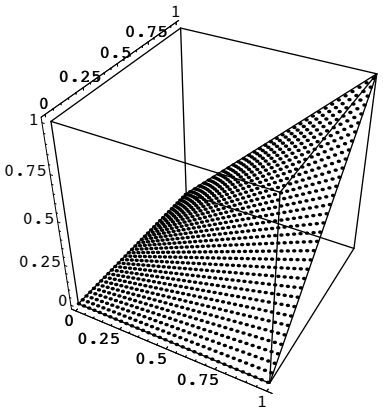


FIG. 2. Shape function for a unit cube

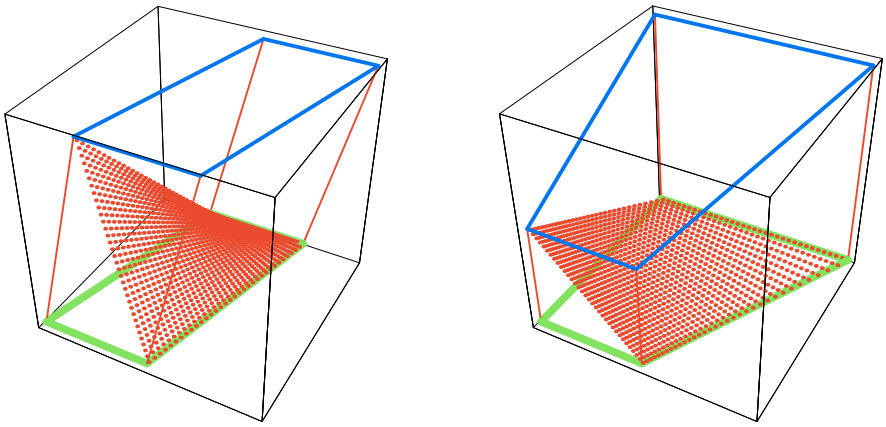


FIG. 3. Shape function from perspective transformations, parametrized coordinates and the Wachspress method respectively

The form of matrix $[\mathbf{A}]$ determines the nature of the transformation. Determining a shape function using parametrized coordinates is equivalent to applying a three dimensional affine transformation. Similarly, Wachspress quadrilateral formulation is equivalent to a two dimensional perspective transform.

Parametrized Coordinates

Let $\{s,t\}$ parametrize the cartesian $\{x,y\}$ coordinates according to equation 5. Let the equation for the thid node be $\phi_c = N_3(x,y)$. Solving for the shape function ϕ_c in terms of x and y. Notice that the resulting equation is a quadratic in ϕ_c:

$$\alpha\,\phi_c^2 - \beta(x,y)\phi_c + \gamma(x,y) = 0 \tag{8}$$

where the coefficients are polynomials in (x,y):

$$\begin{aligned} \alpha &= (\triangle(c,d,a) - \triangle(b,a,d))(\triangle(c,a,b) + \triangle(b,a,d)) \\ \beta(x,y) &= \triangle(\vec{x},a,d)(\triangle(c,a,b) + \triangle(b,a,d)) + \\ &\quad \triangle(\vec{x},b,a)(\triangle(c,d,a) + \triangle(b,a,d)) - \triangle(b,a,d)^2 \end{aligned}$$

$$\gamma(x,y) \quad = \quad \triangle(\vec{x},a,d)\triangle(\vec{x},b,a) \tag{9}$$

The coefficient α is a constant, $\beta(x,y)$ is linear and $\gamma(x,y)$ is quadratic in x and y.

Wachspress convex polygon

A shape function for node-i of a general two dimensional finite element can be constructed according to the following formula (Wachspress 1975). The formulation can be written in compact form (Dasgupta 2000). The node numbering is cyclic, when node-1 is adjacent to node-n and node-m, node-0 = node-n and node-2 = node-m. The order of the numbering is performed according to the right hand rule. A convex element with any number of nodes:

$$N_j(x,y) = \frac{\triangle(j-1,j,j+1)\left(\prod\limits_{\forall k\neq j \text{ or } j-1} \triangle(\vec{x},k,k+1)\right)}{\sum\limits_{i=1}^{n} \triangle(i-1,i,i+1)\left(\prod\limits_{\forall k\neq j \text{ or } j-1} \triangle(\vec{x},k,k+1)\right)} \tag{10}$$

The shape function found for a quadrilateral from this formulation is:

$$\phi_c = \frac{\triangle(b,c,d)\triangle(\vec{x},d,a)\triangle(\vec{x},a,b)}{\left(\begin{array}{c}\triangle(d,a,b)\triangle(\vec{x},b,c)\triangle(\vec{x},c,d)+\triangle(a,b,c)\triangle(\vec{x},c,d)\triangle(\vec{x},d,a)\\ +\triangle(b,c,d)\triangle(\vec{x},d,a)\triangle(\vec{x},a,b)+\triangle(c,d,a)\triangle(\vec{x},a,b)\triangle(\vec{x},b,c)\end{array}\right)}. \tag{11}$$

SIDE-NODE

If a quadrilateral degenerates into an element with a side-node the Wachspress convex polygon method fails. The shape function for the midside node is $N_3(x,y) = 0$. Let point $\{x_c, y_c\}$ be the side-node located a distance u along line $\bar{bd}$:

$$\{x_c, y_c\} = \{x_b, y_b\}(1-u) + \{x_d, y_d\}u. \tag{12}$$

The parametrized coodrinate method produces the follwing coefficients:

$$\begin{aligned} \alpha &= u(1-u) \\ \beta(x,y) &= u\frac{\triangle(\vec{x},a,b)}{\triangle(d,a,b)} + (1-u)\frac{\triangle(\vec{x},d,a)}{\triangle(b,d,a)} - 1 \\ \gamma(x,y) &= \frac{\triangle(\vec{x},d,a)}{\triangle(b,d,a)}\frac{\triangle(\vec{x},a,b)}{\triangle(d,a,b)} \end{aligned} \tag{13}$$

Choosing the branch which satisfies the nodal point values:

$$\phi_c(x,y) = \frac{\beta(x,y) - \sqrt{\beta^2(x,y) - 4\alpha\gamma(x,y)}}{2\alpha}. \tag{14}$$

Test to make sure this side-node formulation satisfies the shape function conditions, equations 2, 3 and 4.

Test side-node shape function

Evaluating at the nodal points, note $0 < u < 1$:

$$\phi_c(x_a,y_a) = \frac{1-\sqrt{1}}{u(1-u)} = 0, \qquad \phi_c(x_b,y_b) = \frac{u-\sqrt{u^2}}{u(1-u)} = 0,$$

$$\phi_c(x_c, y_c) = \frac{2u(1-u) - \sqrt{0}}{2u(1-u)} = 1, \qquad \phi_c(x_d, y_d) = \frac{(1-u) - \sqrt{(1-u)^2}}{2u(1-u)} = 0 \quad (15)$$

Along the boundary, the coefficient $c(x, y)$ is zero valued along the boundaries from point-(a) to point-(b) and from point-(d) to point-(a):

$$\beta(x, y) - \sqrt{\beta(x, y)^2} = 0 \quad \forall\, \beta(x, y) \geq 0 \text{ and } \alpha \neq 0. \quad (16)$$

Along the boundary from point-(b) to point-(d):

$$\frac{(t + u - 2ut) - |t - u|}{2u(1-u)}, \quad (17)$$

the funciton is zero when $t = 0$ and $t = 1$, single valued at $t = u$ and otherwise linear along the boundary.

Within the triangle $\triangle(d, a, b)$ the coefficients α, $\beta(x, y)$ and $\gamma(x, y)$ are bounded. The ratio of areas of a triangle constructed from a point inside a larger triangle and two of its vertices, is less than one and greater or equal to zero if the nodes are in cyclical order. Consequently:

$$0 \leq \phi_c(x, y) \leq 1 \quad \forall\, \{x, y\} \in \mathcal{D}. \quad (18)$$

Solving for the remaining shape functions using the same method as for $\phi_c(x, y)$, find:

$$\phi_a(x, y) = \frac{\triangle(\vec{x}, b, d)}{\triangle(a, b, d)}, \quad \phi_b(x, y) = \frac{\triangle(\vec{x}, d, a)}{\triangle(a, b, d)} - (1-u)\phi_c(x, y)$$
$$\text{and} \qquad \phi_d(x, y) = \frac{\triangle(\vec{x}, a, b)}{\triangle(a, b, d)} - u\phi_c(x, y). \quad (19)$$

The constant and linear fields can be reproduced:

$$\begin{aligned} \phi_a(x, y) + \phi_b(x, y) + \phi_c(x, y) + \phi_d(x, y) &= 1 \\ x_a\phi_a(x, y) + x_b\phi_b(x, y) + x_c\phi_c(x, y) + x_d\phi_d(x, y) &= x \\ y_a\phi_a(x, y) + y_b\phi_b(x, y) + y_c\phi_c(x, y) + y_d\phi_d(x, y) &= y. \end{aligned} \quad (20)$$

Finally, within the triangular domain, the polynomial under the square root is greater or equal to zero:

$$\beta^2(x, y) - 4\alpha\gamma(x, y) \geq 0. \quad (21)$$

According to these shape funciton requirements, the shape function for a midside node is valid.

EXAMPLE

Given a set of nodal points:

$$\{\{-1, 0\}, \{-1/2, 1/4\}, \{1/2, 3/4\}, \{0, 2\}\}, \quad (22)$$

The associated shape functions are:

$$\begin{aligned} \phi_a(x, y) &= \frac{-1 - 4\,x - 4\,y + \sqrt{3}\,\sqrt{-5 + 4\,x\,(1 + 3\,x) + 16\,y}}{6} \\ \phi_b(x, y) &= \frac{11 + 2\,x + 8\,y - 3\,\sqrt{3}\,\sqrt{-5 + 4\,x\,(1 + 3\,x) + 16\,y}}{12} \end{aligned}$$

$$\begin{aligned}\phi_c(x,y) &= \frac{7+10\,x-8\,y+\sqrt{3}\,\sqrt{-5+4\,x\,(1+3\,x)+16\,y}}{12}\\ \phi_d(x,y) &= \frac{-1-x+2\,y}{3}.\end{aligned} \tag{23}$$

The functions are smooth within the triangular domain, see figure 4. The four noded triangle formulation can be extended to any any domain where the shape functions for the parent element, without side-nodes, is known, see figure 5.

FORM OF THE SHAPE FUNCTION

The form of the finite element shape function can not be limited to polygonal form if the domain being described is not triangular or rectangular. Rational polynomials and square roots of polynomials provide a consistent and conformal shape function base for non-concave shapes.

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APPENDIX I. NOTATION

Determinant of matrix $[\mathbf{A}]$: $\ldots\ldots\ldots

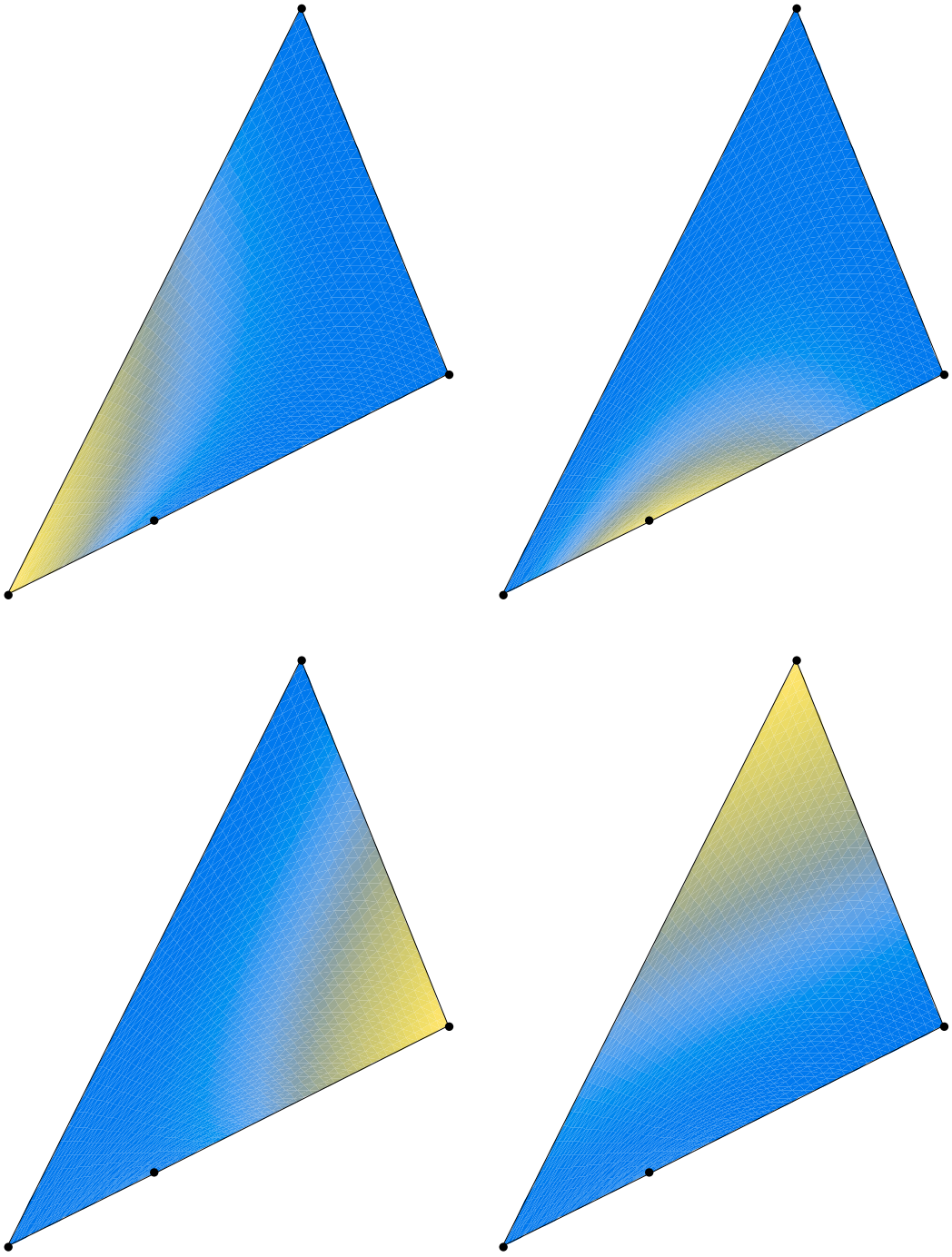


FIG. 4. Smooth shape functions for an element with a side-node

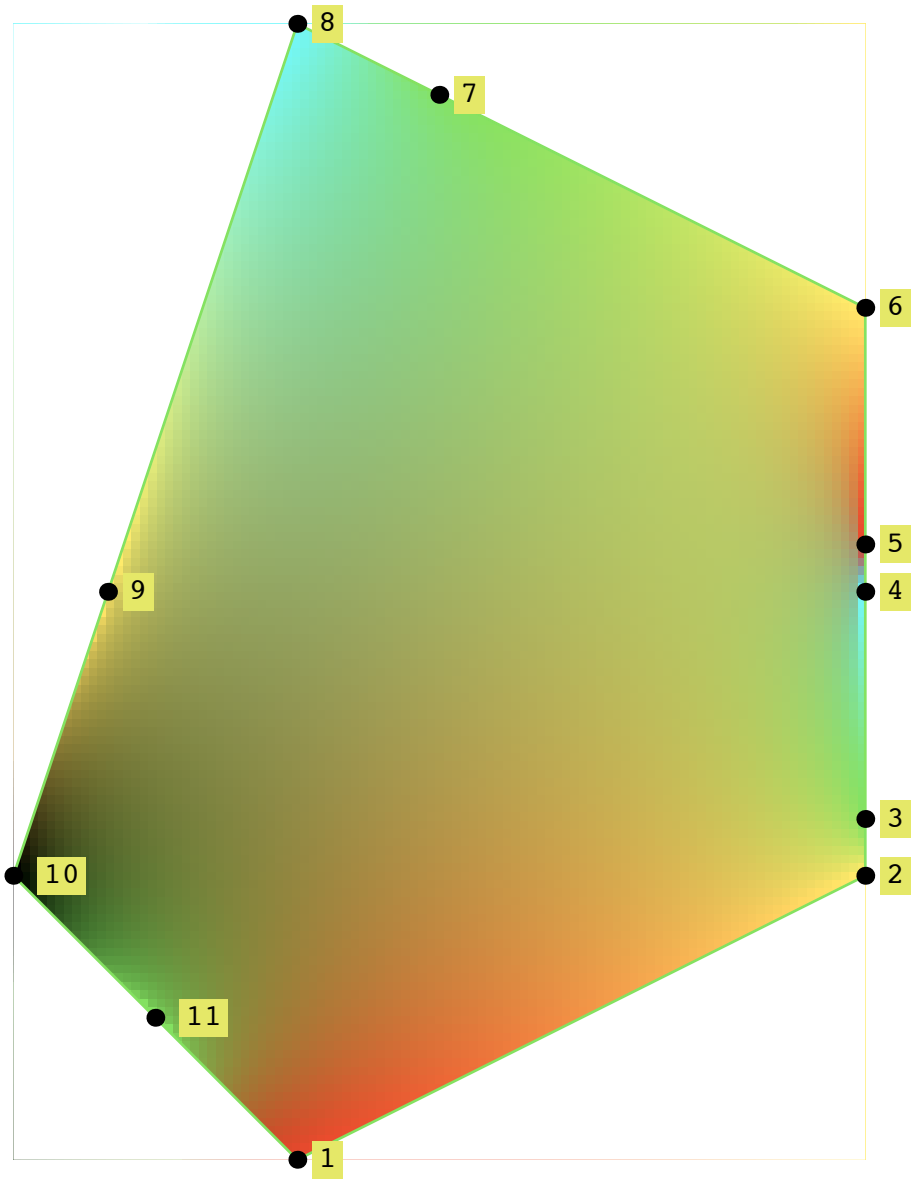


FIG. 5. Smooth shape functions for a pentagonal element with side-nodes