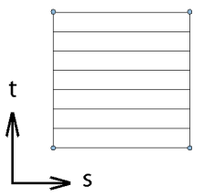
Irrational polynomial **shape functions** in **algebra** and **pictures**

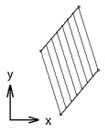
(a) General form of a **non-concave quadrilateral** (Taig, 1961)

$$X(s,t) = X_1(1-s)(1-t) + X_2 s(1-t) + X_3 s t + X_4 t(1-s)$$
$$Y(s,t) = Y_1(1-s)(1-t) + Y_2 s(1-t) + Y_3 s t + Y_4 t(1-s)$$

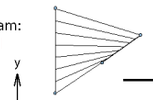
$$\alpha = \mathbf{a}_i, \quad \beta(x,y) = \mathbf{b}_0 + \mathbf{b}_1 x + \mathbf{b}_2 y \quad \textit{and}$$
$$\gamma(x,y) = \mathbf{c}_0 + \mathbf{c}_1 x + \mathbf{c}_2 y + \mathbf{c}_3 xy + \mathbf{c}_4 x^2 + \mathbf{c}_5 y^2$$

$$\alpha\phi^2 - \beta(x,y)\,\phi + \gamma(x,y) = 0$$

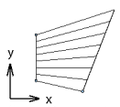




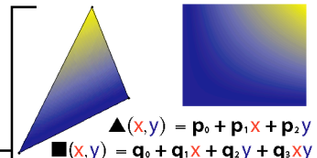
Parallelogram:
Polynomial



Triangle with side node:
Irrational Polynomial

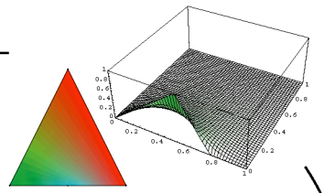


Skew Quadrilateral:
Rational Polynomial



$$\blacktriangle(x,y) = \mathbf{p}_0 + \mathbf{p}_1 x + \mathbf{p}_2 y$$
$$\blacksquare(x,y) = \mathbf{q}_0 + \mathbf{q}_1 x + \mathbf{q}_2 y + \mathbf{q}_3 xy$$

Triangle & **parallelogram**



Any number of **side nodes.**

(b) **Perspective transformations** preserve the linearity of lines (Desargues, 1636).

"If two triangles are in perspective, then the meets at corresponding sides are co-linear."

Homogeneous coordinate representation of the general perspective transformation of a unit square through matrix **[A]**:

$$\langle s,\ t,\ st,\ 1\rangle [\mathbf{A}]_{(4x4)} = \langle X^*, Y^*, \phi^*, \mathbf{h}\rangle$$

$$X = \frac{X^*}{\mathbf{h}}, \quad Y = \frac{Y^*}{\mathbf{h}} \quad \text{and} \quad \phi = \frac{\phi^*}{\mathbf{h}}$$

Transformation *(a)* above is a perspective transformation.

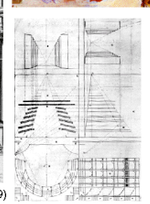
(d) A side node representation for any polygon whose interpolation for the vertex nodes is known:

$$\alpha = u(1-u)$$
$$\beta(x,y) = u\,\phi_d + (1-u)\phi_b - 1$$
$$\gamma(x,y) = \phi_b \phi_d$$

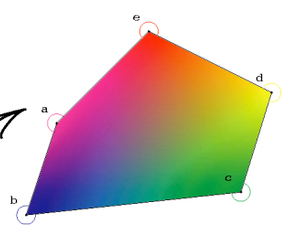
$$\phi_u = \frac{\beta(x,y) - \sqrt{\beta(x,y)^2 + 4\alpha\gamma(x,y)}}{2\alpha}$$







Brother Andrea Pozzo (1642-1709)



Any **n**-sided **convex polygon**

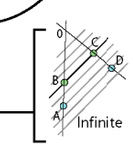
(c) Interpolations within a **convex polygon** with any number of sides can be constructed using tools of **projective geometry** (Wachspress, 1975).

$$\triangle(a,b,c) \triangleq \frac{1}{2}\begin{vmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix}; \quad \text{area of triangle } (a,b,c).$$

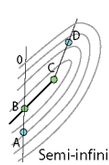
$$\triangle(\vec{x},b,c) \triangleq \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix}; \quad \text{line zero through points } b \& c.$$

$$N_i(x,y) = \frac{\triangle(i-1,i,j-1)\left(\prod_{k\neq i-1,\,k\neq i}\triangle(\vec{x},k,k-1)\right)}{\sum_{i=1}^{n}\triangle(i-1,i,i+1)\left(\prod_{k\neq i,\,k\neq i-1}\triangle(\vec{x},k,k+1)\right)}$$

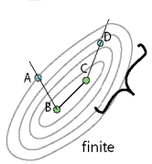
The resulting interpolations are in **rational polynomial** form.



Infinite



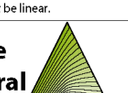
Semi-infinite



finite

(e) A **concave quadrilateral** can not be represented as a perspective transformation of a unit square. The **contour-lines** of the interpolation within a **concave shape** can not be linear.

A **concave quadrilateral**



The interpolation can be constructed as a combination of functions which are **zero** along a **finite stretch** (B,C) and **linear** along the neighboring paths (A,B) and (C,D).

Given a **boundary value problem** of the form:

$$\mathcal{L}[u(x,y)] = 0$$

where $\mathcal{L}[\cdot]$ is a differential operator, and Dirichlet, Neumann or mixed boundary values are given.

An **equivalent variational form** can be constructed, and an **approximate solution** discovered, using **test functions** which satisfy the *geometric* (Dirichlet) *boundary conditions*, but not necessarily the natural (Neumann) conditions (Courant, 1943).

The **test function**, for the *displacement based finite element method*, is constructed as a summation of **nodal values** and functions which **interpolate** those nodal values over the domain.

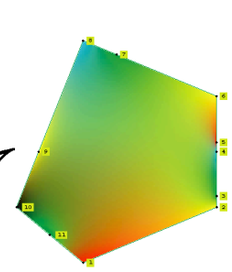
$$\mathbf{u}(x,y) = \{\mathbf{u}\}\cdot\{\mathbf{N}(x,y)\} = \sum_{i=1}^{n} u_i N_i(x,y)$$

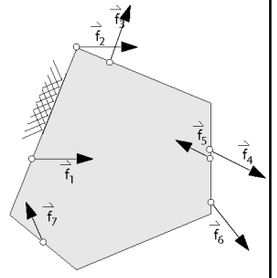
The approximate solution results in a linear equation of the form:

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{f}\}$$

where the matrix **[K]** is an approximation of the operator $\mathcal{L}[\cdot]$, the vector **{u}** is a collection of nodal values, and the vector **{f}** is a collection of forcing functions located at different boundary points.

Smooth **interpolation functions** which satisfy the **essential boundary conditions** allow for approximations within polygons which *do not require tessellation*. If $\mathcal{L}[\cdot]$ is an elliptic differential operator then the behavior of **u**(x,y) is necessarily bounded and smooth within the problem domain. An **interpolation** which captures minimum and maximum properties of the real solution allows for an order of approximation which depends only on the number of nodal values which govern the test function.





TEMPERATURE:

The simplest differential operator is the Laplacian:

$$\nabla^2 u(x,y) = 0$$

Heat distribution over homogeneous domains can be described by the Laplacian. The variational form can be constructed according to a **conductivity matrix**:

$$h_{ij} = \frac{\partial N_i(x,y)}{\partial x}\frac{\partial N_j(x,y)}{\partial x} + \frac{\partial N_i(x,y)}{\partial y}\frac{\partial N_j(x,y)}{\partial y}$$

$$S_{ij} = \int_A h_{ij}\,dA$$

The conductivity matrix **[S]** depends on the interpolating functions **{N(x,y)}**. If the functions are **integrable** and the **derivatives** exist then the matrix can be constructed. *Furthermore, if the functions can, in linear combination, reproduce constant and linear fields, then the constant temperature condition is captured.* The irrational shape functions satisfy the **constancy** and **linearity conditions**.

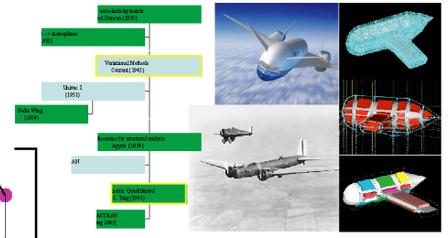
LINEAR ELASTICITY:

For problems, similar to temperature, where the behavior between sources along the **boundary is linear** the **polynomial**, **rational** and **irrational interpolation functions** apply. In *displacement based finite element methods* the displacement fields in the **x** and **y** coordinate directions are assumed according to the interpolation function:

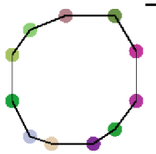
$$\mathbf{u}(x,y) = \{\mathbf{u}\}\cdot\{\mathbf{N}(x,y)\} \ \ \& \ \ \mathbf{v}(x,y) = \{\mathbf{v}\}\cdot\{\mathbf{N}(x,y)\}$$

The **stiffness matrix [K]** resulting from the **variational formulation** similarly depends on the derivatives of the interpolations **{N(x,y)}** and integration over the area of the domain. *Similar to temperature, the rigid body conditions are captured only if the functions, in linear combination, can reproduce constant and linear fields.*

The satisfaction of the **constancy** and **linearity requirement** is essential for the kinematic representation of **thermo-mechanical problems** (Irons, 1980). Any interpolations which satisfy the requirement, and the *linearity and boundedness along the boundary* can be used to construct an approximate solutions.

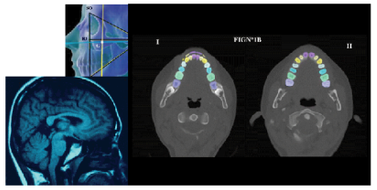


Translation from **CAD** model to **physical model**

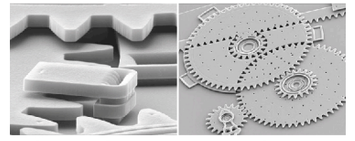




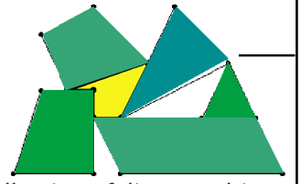
Smooth color **interpolation**



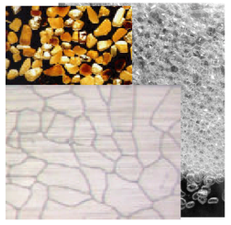
Morphometric analysis



Analysis of sensitive machinery



Collection of discrete objects



References:

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J. C. Taig. Simple and complete by the matrix coplanar point method. Report 5017, English Electric Aviation Report, England, 1961.

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