Kinematic test functions

a general derivation applicable to any simply connected polygon

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Abstract

The solution to a boundary value problem can be approximated using the finite element or boundary element method. The first satisfies the essential boundary conditions and the second the governing field equation. Additional errors are incurred as the result of tessellation and numerical integration. A kinematic test function method which does not require domain tessellation and satisfies the essential boundary conditions can be constructed. A general method which applies to any convex or concave polygon with any number of nodes is presented.

Introduction

Kinematic test functions have been employed effectively in the solution of a great variety of boundary value problems. Courant's Triangle and Taig's Quadrilateral are examples of test functions for triangular and quadrilateral domains respectively [1, 4]. In 1975, E. L. Wachspress developed a general method for deriving test functions on any convex polygonal domain. Each formulation satisfies the linear essential boundary conditions of a boundary value problem exactly and is smooth and bounded in its respective domain. Consequently, each can be used to approximate the solution to a boundary value problem using the variational approximation developed independently by Rayleigh in 1877 and Ritz in 1908 [10]. Each set of functions is linearly independent and reproduces constant and linear fields exactly. Accordingly, shape functions for a $\mathtt{C}^0$ triangular, quadrilateral and general convex finite element approximation can be constructed [11, 14, 12].

The test functions can be derived in general for any domain. Define a function $s_i(\vec{x})$ which is zero and minimum at every data point except for point (i) and every boundary except the adjacent within the domain. A shape function $N_i(\vec{x})$ can be constructed as a combination of $s_i(\vec{x})$ such that the boundary behavior is bounded and the combination of functions $N_i(\vec{x})$ can reproduce constant fields exactly:

$$N_i(\vec{x}) = \frac{k_i s_i(\vec{x})}{\sum_{j=1}^{n} k_j s_j} \quad \text{and} \quad \sum_{i=1}^{n} N_i(x,y) = 1. \tag{1}$$

The resulting set of shape functions $N_i(\vec{x})$ are automatically linearly independent:

$$N_i(\vec{x}_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \tag{2}$$

By construction, the behavior on the boundary is a function only of the $s_j(\vec{x})$ which define the points on the ends of the boundary segments, all other functions $s_k(\vec{x})$ are zero.

$$N_j(\vec{x}_i t + \vec{x}_j(1-t)) = \frac{k_i s_i(\vec{x}_i t + \vec{x}_j(1-t))}{k_i s_i(\vec{x}_i t + \vec{x}_j(1-t)) + k_j s_j(\vec{x}_i t + \vec{x}_j(1-t))} \tag{3}$$

Additional conditions, such as linearity requirements, can can be imposed on the functions $s_i(\vec{x})$.

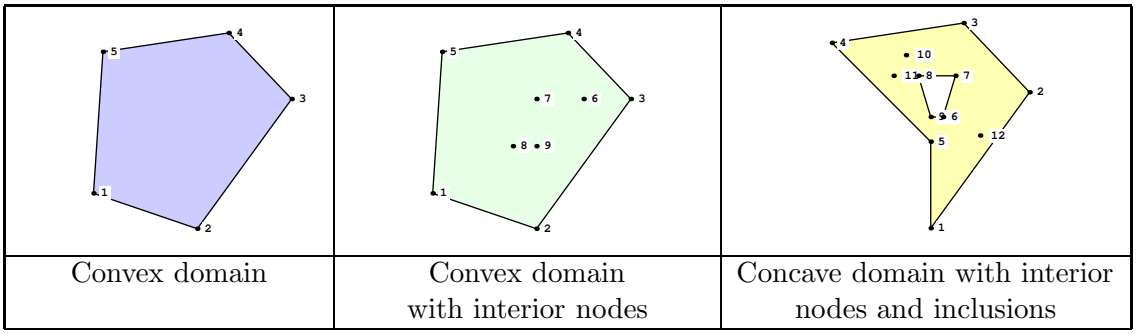
$$s_l(\vec{x}) = \sum_{i\neq l \ \&\ i\neq m}^{n} \frac{s_i(\vec{x}) \triangle(\vec{x}, i, m)}{\triangle(\vec{x}, l, m)} \quad \text{and} \quad s_m(\vec{x}) = \sum_{i\neq l \ \&\ i\neq m}^{n} \frac{s_i(\vec{x}) \triangle(\vec{x}, l, i)}{\triangle(\vec{x}, l, m)} \tag{4}$$

Any set of shape functions $N_i(x,y)$, constructed according to equation (1) with two functions $s_l(x,y)$ and $s_m(x,y)$, defined according to equation (4), satisfies the constant and linearity patch test requirements [4, 11]:

$$\sum_{i=1}^{n} N_i(x,y) = 1, \quad \sum_{i=1}^{n} x_i N_i(x,y) = x \quad \text{and} \quad \sum_{i=1}^{n} y_i N_i(x,y) = y. \tag{5}$$

1 Derivation of interpolations for general polygons

The remaining functions $s_i(x,y)$, whose behavior is not prescribed, can be defined on any polygonal domain as the product of functions which are zero along the boundary segments and at the interior points within the domain.



Convex domain	Convex domain with interior nodes	Concave domain with interior nodes and inclusions

1.1 The Wachspress Convex Element

The simplest set of functions $s_i(x,y)$ are defined as the product of linear functions $r_{ab}(\vec{x})$ which are zero valued and minimum along a line $\bar{ab}$ [12]. For convenience the function can be described as the area of the triangle with nodal points (a), (b) and $(\vec{x})$. If the three points are collinear along the line $\bar{ab}$, the area of the triangle and the value of the function is zero.

$$\hat{r}_{ab}(x,y) = \triangle(\vec{x}, a, b) \tag{6}$$

Any convex domain the boundary can be described by such functions $r_{ab}(x,y)$ as follows:

$$s_i(x,y) = \prod_{i\neq a \ \&\ i\neq b}^{n} \hat{r}_{ab}(x,y) \quad \text{and} \quad N_i(x,y) = \frac{k_i \prod\limits_{\substack{i\neq a \\ i\neq b}} r_{ab}(x,y)}{\sum_{j=1}^{n} k_j \prod\limits_{\substack{j\neq a \\ j\neq b}} r_{ab}(x,y)}. \tag{7}$$

To ensure the function $N_i(\vec{x})$ is linear along the boundary solve equation (3) for $N_j(\vec{x}_i t + \vec{x}_j(1-t)) = t$ and find $k_i = \triangle(i-1, i, i+1)$. To ensure the set of shape functions satisfies the constancy condition: solve equation (4) for $s_m(\vec{x})$ and $s_l(\vec{x})$. For this simplest linear case, the derived functions $s_m(\vec{x})$ and $s_l(\vec{x})$ are of the form assumed for the $s_i(x,y)$, in equation (7a). The details of the derivation are outlined in the reference [2].

1.2 Convex elements with internal nodes

If data points are given at the interior of the domain, and not just the boundary vertices, the definition of the $s_i(x,y)$ can be augmented such that the influence of the interior nodes is included. Conventional methods for adding points within a domain without using meshing include bubble

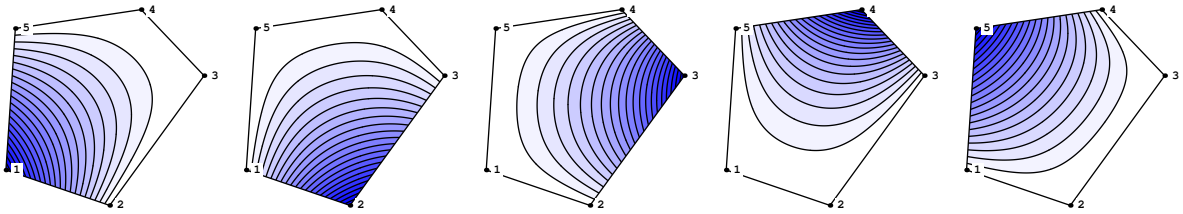


Figure 1: Test function contours on a convex pentagon

functions and mesh-less methods [14]. Bubble functions do not necessarily satisfy rigid body conditions, and mesh-less methods are not conformal [6]. The kinematic test function formulation does not exhibit these pathologies. Let:

$$s_i(x,y) = \left(\prod_{i \neq a \ \& \ i \neq b}^{n} r_{ab}(x,y) \right) \times \left(\prod_{j=1}^{p} l_j(x,y) \right) \quad \text{where} \quad l_j(\vec{x}) = \sqrt{(x-x_j)^2 + (y-y_j)^2} \tag{8}$$

and p is the number of internal nodes located at the points $\{x_j, y_j\}$. The test functions for the internal points are:

$$s_j(x,y) = \left(\prod_{\text{all}}^{n} r_{ab}(x,y) \right) \times \left(\prod_{j \neq k}^{p} l_k(x,y) \right) \tag{9}$$

The linearity of the edges is not disturbed. The linearity requirement can be enforced by reevaluating two of the nodal influence functions $s_m(x,y)$ and $s_l(x,y)$ for two adjacent nodes on the convex boundary using equation (4).

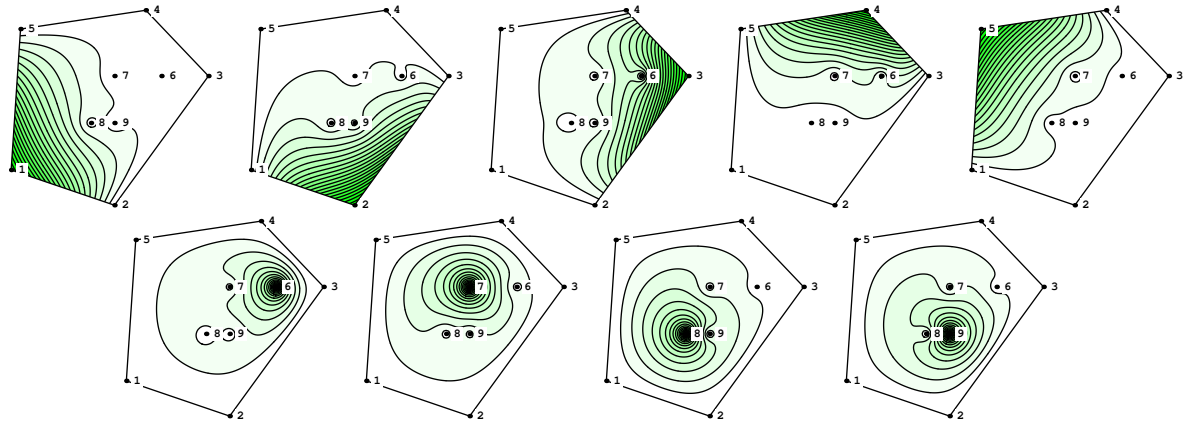


Figure 2: Test function contours on a convex pentagon with interior points

1.3 Simply connected polygons with concavities and inclusions

The linear functions $\hat{r}_{ab}$ in equation (6) applied to a reentrant boundary or the boundary of an inclusion would prescribe a minimum zero value in the interior of the domain. Consequently, a higher order formulation is required. Leaving away (x,y) for convenience, a general function which is linear along incoming angle α and exiting angle β is:

$$\bar{r}_{ab} = \frac{(A\sin(\alpha) + B\sin(\beta))((l_{b\vec{x}} + l_{a\vec{x}})^2 - l_{ab}^2)(l_{ab}/2)}{(l_{b\vec{x}} + l_{a\vec{x}})(A+B) - l_{ab}(A\cos(\alpha) + B\cos(\beta))} \tag{10}$$

where

$$A = \left(l_{b\vec{x}}^2 + l_{ab}^2 - 2l_{ab}l_{b\vec{x}}\cos\beta\right) - l_{a\vec{x}}^2 \quad \text{and} \quad B = \left(l_{a\vec{x}}^2 + l_{ab}^2 - 2l_{ab}l_{a\vec{x}}\cos\alpha\right) - l_{b\vec{x}}^2.$$

The angles can be defined in terms of the boundary nodes:

$$\cos(\alpha) = \frac{l_{ab}^2 + l_{da}^2 - l_{bd}^2}{2l_{ab}l_{da}}, \quad \sin(\alpha) = \frac{2|\triangle(d,a,b)|}{l_{da}l_{ab}}, \quad \cos(\beta) = \frac{l_{ab}^2 + l_{bc}^2 - l_{ca}^2}{2l_{ab}l_{bc}} \quad \text{and} \quad \sin(\beta) = \frac{2|\triangle(a,b,c)|}{l_{bc}l_{ab}}$$

Where, the nodes along the boundary, in order, are (d, a, b, c) and the function r_{ab} is zero valued along $\bar{ab}$, and linear along $\bar{da}$ and $\bar{bc}$. Consequently, it can be used to describe the boundary of a reentrant portion or inclusion in the polygon.

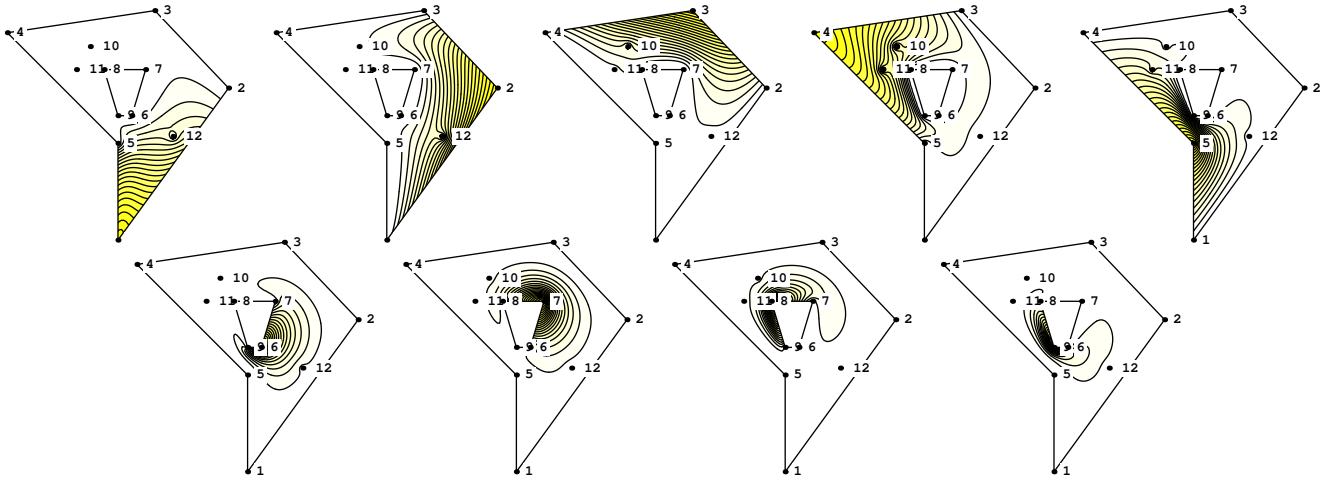
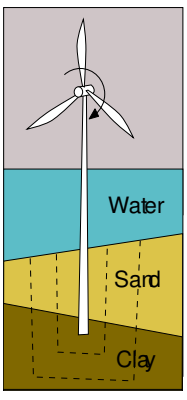


Figure 3: Test function contours on a concave polygon an inclusion and interior points

2 Conclusion: extensions and applications

Established methods for extending Wachspress type element shape functions can also be applied to these more general shapes. Nodes on the boundary can be added while preserving the linearity of the edges and the smoothness of the domain [7]. The formulation can be extended to describe a simply connected polytope of any dimension [13]. Since the formulation satisfies the constancy and linearity conditions it can be extended, using the multinomial theorem, to be a basis function of any order [5, 3]. Similar formulations applicable to curved sides have been developed [12]. A general method for polytopes with curved sides is the subject of ongoing research.



An analysis of an irregular polygonal domain can be performed using domain decomposition and local shape functions. The presented method avoids the heuristic choices and built-in instabilities associated with the conventional approach. The kinematic test function formulation applies to any simply connected polygon. Linear essential boundary conditions can be reproduced exactly.

The presented kinematic test functions will be employed in applications where domain tessellation is incurs unacceptable errors. Wachspress convex element shape functions have already been applied successfully to biomedical problems including skull growth and heart function analyses [8]. A first application of the presented general test functions will be the study of offshore windmill foundations, see figure left [9]. Specifically, a concave shaped element is necessary for stable wave propagation approximations, using for example the cloning algorithm.

Notation

$\triangle(a,b,c) = \frac{1}{2}\begin{vmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix}$ ... Area of the triangle with vertices located at points a,b and c.

$\triangle(\vec{x},b,c) = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix}$ ... Area of the triangle with vertices located at points $\{x,y\}$,b and c.

$l_{ab} = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$ ..... Length of a line with end points at a and b.

$l_{a\vec{x}}(x,y) = \sqrt{(x - x_a)^2 + (y - y_a)^2}$ ..... Length of a line with end points at a and $\{x,y\}$.

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