Random Processes

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Abstract

Random processes are analyzed using statistical and probabilistic methods. If the process is stationary a fourier transform can simplify a random looking function into a collection of coefficients, thus transforming the process to the frequency domain.

Random Process: A random process does not have a predominant amplitude or frequency.

1 Introduction

1.1 References

- Random Vibrations **??**
- pranzen **??**
- papoulis **??**
- Class notes R. Vaicaitis EM E6220 Fall 2000

1.2 Class

Class meets from $4:10-6:40$ on tuesday. The midterm is worth 200 points and the final worth 300 points, homework is not collected for a grade.

1.3 General Applications

Random processes occur in nature. The Aeronautical Engineers must take random gust componenets of lift drag and aeronautical moment into airplane design. Maneuver loads also applear random over time, their effect is even stronger than that of normal gusts and turbulence. The prediction of cabin noise is based on the understanding of random jet noise and turbulent surface folow. The landing gear of an airplane must also withstand a random process, however the analysis is further complicated since the process is *non-stationary*. Civil engineers also consider random processes when designing buildings and bridges to wtihst and

wind earthquake and ocean loads. Some design can be achived while considering only *pseudo-random* forces. Mechanical engineers consider vehical response and mechanical vibrations. Electrical engineers must consider the effect of random processes in signal analysis and communications. Much of the bio-engineering field is effected by randomness: accoustics, noise, speach inference, heart beat, turbulence in blood flow and brain waves.

1.4 Random analysis

- Measurements
  - Data Aquisiton
  - Data Reduction
  - Data Interpolation
- Statistical Inference
  - Bounds
  - Tolerances
  - Confidence limits
  - Risk
- Probabilistic (Random) Models ← focus of class

2 Random Variables, Probability Distributions and Averages

2.1 Single Random Variables

Random Variable (χ): is a function such that for every x both real and complex there exists a probability of $\chi = x$.

Discrete Random Variable: is a variable whose sample space consists of integer values $n \in 1, 2, 3, \ldots$.

Probability Mass Function: the probability that $\chi = x$.

$$0 \leq P_\chi(x) \leq 1 \tag{1}$$

$$\sum_{allx} P_\chi(x) = 1 \tag{2}$$

2.1.1 Discrete Distribution

Bernouli Distribution: The random variable κ is definded within the sample space: $S = \{0, 1\}$.

$$P_\kappa(k) = \begin{cases} p & k = 1 \\ 1 - p & k = 0 \\ 0 & else \end{cases} \tag{3}$$

Poisson Distribution: A random variable definde in the sample space $S = \{0, 1, 2, , 3, \ldots\}$. It may be used to count the number of occurences of some event in a given time interval $(0, t)$

$$P_N(n) = \begin{cases} \frac{(\lambda t)^n e^{-\lambda t}}{n!} & n = 0, 1, 2, \ldots \\ 0 & else \end{cases} \tag{4}$$

Example: What is the probability of a 7,4 or 12 in a throw of a pair of dice.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(\chi = 7) = \frac{1}{6} \tag{5}$$

$$P(\chi = 4) = \frac{1}{12} \tag{6}$$

$$P(\chi = 12) = \frac{1}{36} \tag{7}$$

2.1.2 Continuous Random Variables and Probability Density Functions

Continuous random variable: is defined in the sample space $S \in \Re$.

Probability: the chance that the random variable is a given value $\chi = x$.

$$P(a < x < b) = \int_a^b f_\chi(x)\mathrm{d}x \tag{8}$$

Cumulative density function:

$$F_\chi(x) = P_\chi(\chi \leq x) \tag{9}$$

$$F_\chi(-\infty) = 0 \quad \text{and} \quad F_\chi(\infty) = 1 \tag{10}$$

Probability density function:

$$f_\chi(x) = \lim_{\Delta x \to 0} \frac{F_\chi(x + \Delta x) - F_\chi(x)}{\Delta x} \tag{11}$$

$$f_\chi(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_\chi(x) \quad \text{and} \quad \int_{-\infty}^{\infty} f_\chi(x)\mathrm{d}x = 1 \tag{12}$$

2.1.3 Distributions

Uniform: A random variable χ is uniformly distributed over an interval $a \leq x \leq b$.

$$f_\chi(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & else \end{cases} \tag{13}$$

Gaussian:

$$f_\chi(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \mathrm{e}^{\frac{-(x-\mu)^2}{2\sigma_\chi}} \tag{14}$$

Rayleigh: used for describing the peak values of a random process.

$$f_\chi(x) = \frac{x}{\sigma_x} \mathrm{e}^{-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2}; \qquad x > 0 \tag{15}$$

2.1.4 Expected Values

Expected values are a alculation of averages. The n^{th} statistical moment is:

$$E[x^n] = \int_{-\infty}^{\infty} x^n f_\chi x \mathrm{d}x \tag{16}$$

The rist moment is the mean, the second moment is the mean square, the third is skewness and the fourth kurtosis.

Variance:

$$\sigma_\chi^2 = E[(x-\mu_\chi)^2] = E[x^2] - 2\mu_\chi E[x] + \mu_\chi^2 = E[x^2] - \mu_\chi^2 \tag{17}$$

Standard of Deviation:

$$\sigma_\chi = \sqrt{E[x^2] - \mu_\chi} \tag{18}$$

Example: Consider χ has a uniform probability density function. Given:

$$f_\chi(x) = \begin{cases} \frac{1}{2\pi} & 0 \leq x \leq 2\pi \\ 0 & else \end{cases} \tag{19}$$

Determine $\mu_x, E[x^2], \sigma_\chi$.

$$\mu_\chi = \int_{-\infty}^{\infty} x f_\chi(x) \mathrm{d}x = \pi \tag{20}$$

$$E[x^2] = \int_0^{2\pi} x^2 f_\chi(x) \mathrm{d}x = \frac{4}{3}\pi^2 \tag{21}$$

$$\sigma_X = \frac{\pi}{\sqrt{3}} \tag{22}$$

2.2 Jointly Distributed Random Variables

2.2.1 Joint Probability Distribution

Let $X(x)$ and $Y(y)$ be two random variables.

Joint Probability Distribution:

$$f_{XY} = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \tag{23}$$

Single distributions can be recoved from the joint distribution.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}x \qquad f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}y \tag{24}$$

Cumulative Density Distribution:

$$F_{XY} = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(\zeta,\eta)\mathrm{d}\zeta\mathrm{d}\eta \tag{25}$$

Normalization condition:

$$F_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(\zeta,\eta)\mathrm{d}\zeta\mathrm{d}\eta = 1 \tag{26}$$

Conditional Distributions: The conditional probability density function of X given the outcome of the random variable Y.

$$f_{X|Y}(x|Y_{=y}) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{27}$$

Statistical Independence: X and Y are independent.

$$f_{X|Y}(x|Y_{=y}) = f_X(x) \qquad f_{XY} = f_X(x)f_Y(y) \tag{28}$$

2.2.2 Expected Values, Covariance and Correlation Functions

Joint Statistical Moments:

$$E[X^n Y^m] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X^n Y^m f_{XY}(x,y)\mathrm{d}x\mathrm{d}y \tag{29}$$

Covariance:

$$\sigma_{xy} = E[(x-\mu_x)(y-\mu_y)] = E[XY] - \mu_x\mu_y \tag{30}$$

Correlation Coefficient:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \qquad -1 \le \rho_{XY} \le 1 \tag{31}$$

if X and Y are independent then $\rho_{XY} = 0$.

2.2.3 Joint Distributions

The bivariate normal gaussian:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{XY}^2}} \mathrm{e}^{-\frac{1}{2\left(1-\rho_{XY}^2\right)}\left\{\frac{(x-\mu_x)^2}{\sigma_x^2} - \rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_x\sigma_y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right\}} \tag{32}$$

Multivariate Jointly Distributed Normal Random Variables:

$$f_{X_1X_2\ldots X_n}(x_1,x_2,\ldots,x_n) = \frac{1}{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\left|[S]^{\frac{1}{2}}\right|} \exp\left[-\frac{1}{2}(\vec{x}-\vec{m})^T[S]^{-1}(\vec{x}-\vec{m})\right] \tag{33}$$

$S = E[X_iX_j]$: symmetric covariance matrix.
$x_i - m_i = x_i - \mu_i$: normalization.

2.2.4 Functions of Random Variables:

$Y = g(X)$ since X is a random variable Y is also a random variable. The statistical properties of Y depend on X.

For functions whose inverse are also functions.

$$f_Y(y) = f_X(x)\left|\frac{\partial x}{\partial y}\right| \quad a \le x \le b,\ \min g(x) \le y \max g(x) \tag{34}$$

For functions whose inverse have n mappings.

$$f_Y(y) = n \times f_X(x)\left|\frac{\partial x}{\partial y}\right| \tag{35}$$

Example:
(1) Given $f_X(x)$ in $x \in [0,1]$ and $Y = \frac{X}{2}$.

$$\left|\frac{\partial x}{\partial y}\right| = 2 \quad f_Y(y) = f_X(x)\cdot 2 = 2f_X(2y) \quad y \in [0,\frac{1}{2}] \tag{36}$$

(2) Given $Y = \sin X$
where $f_X(x)$ is a uniform distribution between $[0, 2\pi]$.

$$\left|\frac{\partial x}{\partial y}\right| = \frac{\mathrm{d}}{\mathrm{d}y}\sin^{-1}(y) = \frac{1}{\sqrt{1-y^2}} \tag{37}$$

$$f_Y(y) = 2 \times \frac{1}{2\pi}\frac{1}{\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}} \quad y \in [-1,1] \tag{38}$$

(3) Given $Y = X^2$.

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{\partial}{\partial y}\left(P[-\sqrt{y} \leq x \leq \sqrt{y}]\right) = \frac{\partial}{\partial y}\int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\mathrm{d}x \quad (39)$$

Solve using Leibnitz Rule:

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \quad (40)$$

2.2.5 Central Limit Theorem:

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with the means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1, \sigma_2, \ldots \sigma_n$.

Let S_n be the sum of the sequence:

$$S_n = \sum_{i=1}^{n} X_i \quad M_{sn} = \sum_{i=1}^{n} \mu_i \quad \sigma_{sn} = \sum_{i=1}^{n} \sigma_i^2 \quad (41)$$

As $n \to \infty$ the standardized variable of S_n. By definition the mean is 0 and the variance is 1.

$$Z_n = \frac{S_n - \mu_{sn}}{\sigma_{sn}} \quad (42)$$

The standardized variable has the following normal distribution:

$$f_{zn}(z) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}z^2} \quad z \in [-\infty, \infty] \quad (43)$$

For any individual distribution of X_i. The distribution of the sum converges to a normalized gaussian distribution.

3 Random Process

3.1 Introduciton

Random Process: $X(t)$ is an ensemble of time functions that can be characterized statistically. It is a parametered family of random variables.

$$\{X(t) \in \Omega, t \in T\} \quad (44)$$

is a random process defined in state space Ω that evolves with respect to the index set T.

Example: Time Realizations
A: $X_1(t_1) + X_2(t_1), \ldots, X_n(t_1)$
B: $X_1(t_2) + X_2(t_2), \ldots, X_n(t_2)$
A and B are random variables.

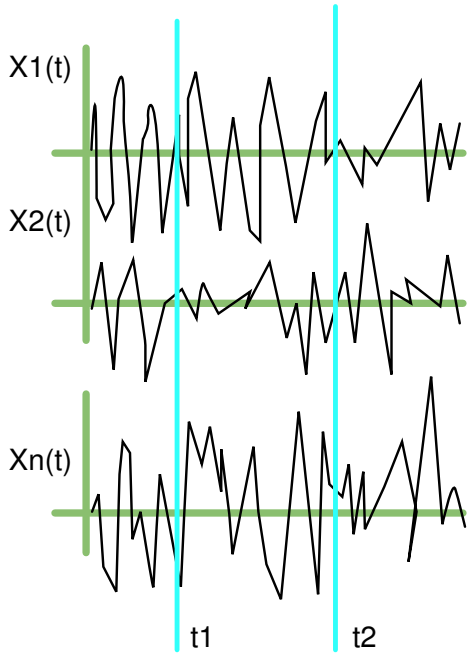


Figure 1: Time Realizations

Random processes can be specified by a probability density function of increasing orders of completeness. The n^{th} order is given as below:

$$f_{X(t_1)X(t_2)\ldots X(t_n)}(x_1, x_2, \ldots, x_n) \tag{45}$$

Characteristic functions, statistical moment functions and others can be used to characterize random processes.

Strongy Homogenous: the complete set of probability functions is independent of shift of the parametric origin $t_i \rightarrow t_i + a$.

Weakly Homogenous: the first two orders of probability funcitons are independent of a parametric origin shift $t_i \rightarrow t_i + a$.

Stationary Process: Homogenous process.

3.2 Ensemble Averages:

The realizations of random processes $X(t)$ is called the ensemble. Statistical moments can be calculated accordingly.

Mean:

$$\mu_x(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) \mathrm{d}x \tag{46}$$

Variance:

$$\sigma_x^2(t) = E[X(t) - \mu_X(t)^2] = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_{X(t)} x \mathrm{d}x \tag{47}$$

Joint: measure of processes at two times $X(t_1)$ with $X(t_2)$ is the correlation function.

$$R_x(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1 X_2 f_{X(t_1)X(t_2)}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \tag{48}$$

also called the **Autocorrelation Function**. It is always positive definite.

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_j \alpha_k R_x(t_1, t_2) \geq 0 \quad \forall a_j \in \Re \tag{49}$$

Autocovariance:

$$\begin{aligned} \sigma_{xx}(t_1, t_2) &= E[\{X(t_1) - \mu_x(t_1)X(t_2) - \mu_x(t_2)\}\{x(t_2) - \mu_x(t_2)\}] \\ &= R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2) \end{aligned} \tag{50}$$

Autocorrelation Coefficient:

$$\rho_{xy}(t_1, t_2) = \frac{\sigma_{xx}(t_1, t_2)}{\sigma_x(t_1)\sigma_x(t_2)} \tag{51}$$

Example: Let $X(t) = \mathbf{A}\cos wt$ where $\mathbf{A}$ is a random variable.
Given a probability density function of $\mathbf{A}$

$$\begin{aligned} \mu_x(t) &= E[A]\cos wt = \mu_A \cos wt \quad \text{if} \quad \mu_A \neq 0 \\ \sigma_X^2(t) &= (E[A^2\cos^2 wt] - \mu_A^2 \cos^2 wt) = \sigma_A^2 \cos^2 wt \\ R_x(t_1, t_2) &= E[X(t_1)X(t_2)] = E[A^2]\cos wt_1 \cos wt_2 \\ &= (\sigma_A^2 + \mu_A^2)\cos wt_1 \cos wt_2 \\ \sigma_{XX} &= R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2) \\ &= (\sigma_A^2 + \mu_A^2)\cos wt_1 \cos wt_2 - \mu_A^2 \cos wt_1 \cos wt_2 \\ &= (\sigma_A^2)\cos wt_1 \cos wt_2 \\ \sigma_{XX}(t_1, t_2 = t_1) &= \sigma_A^2 \cos wt = \sigma_X^2 \quad \text{at} \quad t_1 = t_2 \end{aligned} \tag{52}$$

Example: Given a random process $X(t) = A\cos(wt - \Phi)$ where Φ is a random variable, and A and w are deterministic. Assume a uniform distribution $f_\Phi(\phi) = \frac{1}{2\pi} \quad \forall \phi \in [0, 2\pi]$.

$$\begin{aligned}
\mu_x &= E[X(t)] = \int_{-\infty}^{\infty} X(t) f_X(x) \mathrm{d}x \\
&= \int_0^{2\pi} A\cos(wt - \Phi) f_\Phi(\phi) \mathrm{d}\phi \\
&= \frac{A}{2\pi} \sin(wt - \Phi)|_0^{2\pi} = 0 \\
\sigma_x^2 &= E[(X(t) - \mu_X)^2] = E[X^2] - \mu_X^2 \\
&= \int_0^{2\pi} A^2 \cos^2(wt - \Phi) f_\Phi(\phi) \mathrm{d}\phi \\
&= A^2 \frac{2\pi}{2} \frac{1}{2\pi} = \frac{A^2}{2}
\end{aligned} \tag{53}$$

3.3 Stationary Processes

For a stationary random process the probability structure is invarient to a shift of the time origin. Consequently:

$$\mu_x(t) = \mu_X = \text{const.} \tag{54}$$

$$\sigma_x^2(t) = \sigma_X^2 = \text{const.} \tag{55}$$

Thus the autocorrelation function depends only on the time change $t_2 - t_1$ not the actual times t_1 and t_2.

$$R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau) \quad \text{where} \quad \tau = t_2 - t_1 \tag{56}$$

$$R_X(\tau) = R_x(t, t + \tau) = E[X(t) \cdot X(t + \tau)] \tag{57}$$

$$\sigma_{XX} = R_X(\tau) - \mu_X \quad \text{and} \quad \rho_{XX}(\tau) = \frac{\sigma_X X(\tau)}{\sigma_X^2} \tag{58}$$

Properties of correlation functions R_X include:

- $R_X(\tau) = R_X(-\tau)$ —symmetry
- $|R_X(\tau)| \leq R_X(0)$ —boundedness
- $R_X(0) = \sigma_X^2$
- $\lim_{\tau \to \infty} R_X(\tau) = \mu_X$ for non periodic functions. —limit

Examples of Correlation functions include: $R_X(\tau) = \mathrm{e}^{-a|\tau|}$, $R_X(\tau) = \mathrm{e}^{-a\tau^2}$, and $R_X(\tau) = \mathrm{e}^{-a\tau^2} \cos wt$.

Example: Consider a stationary random process $X(t) = \sin(wt + \Phi)$ where A and w are constants.

$$\begin{aligned} R_X(\tau) &= E[X(t)X(t+\tau)] \\ &= \int_0^{2\pi} A\sin(wt+\Phi)A\sin(wt+\tau+\Phi)f_\Phi(\phi)\mathrm{d}\phi \\ &= \frac{A^2}{2\pi}\int_0^{2\pi}\sin(2t+\tau)\sin(wt+\tau+\Phi)\mathrm{d}\phi \\ &= \frac{A^2}{2}\cos w\tau \end{aligned} \tag{59}$$

This answer satisfies conditions $1-3$ and condition 4 is violated since the function is periodic.

3.3.1 Cross-Correlation Function

For two stationary random processes $X(t)$ and $Y(t)$:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] \tag{60}$$

This is a function of the combined distribution which is hard to find.

Properties include:

- $R_{XY}(\tau) = R_{YX}(\tau)$ —symmetric
- $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$ —bounded
- $\lim_{\tau\to\infty} R_{XY}(\tau) = \mu_x\mu_y$ except for periodic —limit

3.3.2 Ergodic Processes

The mean and variance of a stationary ergodic process can be computed using temporal averaging instead of ensemble averaging. Thus, when random process is ergodic in the mean:

$$E[X(t)] = \langle X(t)\rangle = \mu_x = \frac{1}{T}\int_0^T X(t)\mathrm{d}t. \tag{61}$$

This occurs if and only if $E[X(t)] = \text{const}$, $E[X(t)X(t-\tau)] = R_X(\tau)$ and $\lim_{T\to\infty}\int_0^T R_X(\tau)\mathrm{d}\tau = 0$ (except for periodic processes).

When a random process is ergodic in the correlation:

$$\begin{aligned} E[X(t)X(t+\tau)] &= R_X(\tau) = \langle X(t)X(t+\tau)\rangle \\ &= \lim_{\tau\to\infty}\frac{1}{T}\int_0^\tau X(t)X(t+\tau)\mathrm{d}t \end{aligned} \tag{62}$$

At $\tau = 0$:

$$E[X^2(t)] = \sigma_X^2 = \frac{1}{T}\int_0^T X^2(t)\mathrm{d}t \tag{63}$$

This occurs if and only if: $E[X(t)X(t+\tau)] = f(\tau)$, $S_X(\tau,\mu) = E\{[X(t)X(t+\tau) - R_X(\tau)][X(t+\mu)X(t+\tau+\mu) - R_X(\tau)]\} = g(\tau)$, and $\lim_{t\to\infty} \frac{1}{T}\int_0^T S_X(T,\mu)\mathrm{d}\mu = 0$.

Examples:
$X(t) = C$ with C random, is stationary but not ergodic.
$X(t) = Bt$ with B random,is non stationary but ergodic.
$X(t) = A\cos wt$ with A random, is stationary but not ergodic.
$X(t) = A\cos(wt + \Phi)$ with Φ random, is stationary and ergodic.

3.3.3 Gaussian Process

A random process for which the joint distribution of $X(t_1)X(t_2)\dots X(t_n)$ is a joint normal distribution is a Gaussian or normal process. Just as gaussian random variables are completely characterized by their mean and variance, the gaussian random process is completely characterized by its mean and auto correlation functions. The linear combination of a gausian process is also a gaussian process.

3.4 Derivation and Integration of Stationary Random Processes

3.4.1 Derivation

Given a stochastic (random) process $X(t)$ its derivative is $\dot{X}(t) = \frac{\mathrm{d}X}{\mathrm{d}t}$. The derivative of the auto correlation function is a cross correlation:

$$\frac{\mathrm{d}}{\mathrm{d}t}R_X(\tau) = \frac{\mathrm{d}}{\mathrm{d}t}E[x(t)X(t+\tau)] = E[X(t)\dot{X}(t+\tau)] = R_{X\dot{X}}(\tau) \tag{64}$$

Similarly, $\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}R_X(\tau) = R_{X\ddot{X}}(\tau)$ again a cross correlation.

The variation τ is arbitrary so it can be rewritten $\mu = t + \tau$ and $t = \mu - \tau$.

$$E[X(t)\dot{X}(t+\tau)] = E[X(t-\tau)\dot{X}(t)] \tag{65}$$

now $\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}R_X(\tau) = -R_{\dot{X}\dot{X}}(\tau)$. which is an auto correlation.

If there are no discontinuities in the auto correlation function $R_{X\dot{X}}(0) = 0$. Thus $X(t)$ and $\dot{X}(t)$ are uncorrelated or orthogonal at any given time t.

3.4.2 Integration

For a random process $X(t)$

- Converting a random process into a random variable

$$z = \int_a^b X(t)\mathrm{d}t \tag{66}$$

- from random process to random process

$$z(t) = \int_a^t X(s)\mathrm{d}s \quad \text{and} \quad \mu_z(T) = E[z(t)] = \int_a^t \mu_x(s)\mathrm{d}s \tag{67}$$

$$R_{XZ}(t_1 t_2) = \int_a^{t_2} R_{XX}(t_1 s_2)\mathrm{d}s_2 \tag{68}$$

$$R_{XX}(t_1 t_2) = E[z(t_1)z(t_2)] = \int_0^{t_1}\int_0^{t_2} R_{XX}(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2 \tag{69}$$

- Transforming a process

$$z(\eta) = \int_a^b X(t)g(t,\eta)\mathrm{d}t \quad \text{and} \quad \mu_z = \int_a^b X(t)g(t,\eta)\mathrm{d}t \tag{70}$$

$$R_{ZZ}(\eta_1, \eta_2) = \int_a^b\int_a^b R_{XX}(t,s)g(t,\eta)g(s,\eta)\mathrm{d}t\mathrm{d}s \tag{71}$$

Example:
given $X(t)$ is a stationary random process

$$\mu_X = 0 \quad R_X(\tau) = e^{-\alpha|\tau|} \quad Z(t) = \int_0^t x(s)\mathrm{d}s \quad \mu_X = 0 \tag{72}$$

Determine:

$$\begin{aligned} R_{XZ}(t_1, t_2) &= \int_0^{t_2} R_{XX}(t_1, s_2)\mathrm{d}s_2 \\ &= \int_0^{t_2} R_{XX}(t_1 - s_2)\mathrm{d}s_2 \\ &= \int_0^{t_2} e^{-\alpha(|t_1|)}\mathrm{d}s_2 \end{aligned} \tag{73}$$

For $t_2 \leq t_1$

$$\begin{aligned} R_{XZ}(t_1, t_2) &= \int_0^{t_1} \mathrm{e}^{-\alpha(t_1 - s_2)}\mathrm{d}s_2 \\ &= \frac{e^{-\alpha(t_1 - t_2)} - e^{-\alpha t_1}}{\alpha} \end{aligned} \tag{74}$$

For $t_2 \geq t_1$

$$\begin{aligned} R_{XZ}(t_1, t_2) &= \int_0^{t_1} e^{(-\alpha t_1 - \alpha s_2)} + \int_{t_1}^{t_2} \mathrm{e}^{\alpha t_1 - \alpha s_2} \\ &= \frac{2 - \mathrm{e}^{-\alpha t_1} - \mathrm{e}^{-\alpha(t_2 - t_1)}}{\alpha} \end{aligned} \tag{75}$$

Usefull Properties of Random Variables

Unit Gaussian Random Variable:

$$\mu = 0 \quad \sigma_x = 2 \quad P_U(u) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-\mu^2}{2}} \mathrm{d}u \tag{76}$$

Thus

$$E[u^n] = \int_{-\infty}^{\infty} \frac{u^n}{\sqrt{2\pi}} \mathrm{e}^{\frac{-u^2}{2}} \mathrm{d}u = \begin{cases} 0 & n \in \{\text{odd numbers}\} \\ \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) & n \in \{\text{even numbers}\} \end{cases} \tag{77}$$

Joint Moments of Gaussian Random Variables or Random Processes:

Consider a random variable X_j with $u_j = 0$.

$$E[X_j] = 0 \quad E[X_j^2] = \sigma_{x_j}^2 \quad E[X_j X_k] = \rho_{x_j x_k} \tag{78}$$

$$\begin{aligned} E[X_i X_j X_k] &= E[X_k] E[X_i X_j] \ldots = 0 \quad \text{and} \\ E[X_i X_j X_k X_l] &= E[X_i X_j] E[X_k X_l] + E[X_i X_l] E[X_k X_j] + E[X_i X_k] E[X_j X_l] \\ &\quad + E[X_i] E[X_j X_k X_l] \ldots \\ &= \rho_{x_i x_j} \rho_{x_k x_l} + \rho_{x_i x_l} \rho_{x_j x_k} + \rho_{x_i x_k} \rho_{x_j x_l} \end{aligned} \tag{79}$$

Example:

$$P = \frac{1}{2} \rho V^2 \tag{80}$$

$$E[P^2] = \left(\frac{1}{2}\rho\right)^2 E[V^4] = \left(\frac{1}{2}\rho\right)^2 3\sigma_v^4 \tag{81}$$

In general:

$$E[X_1 X_2 \ldots X_{2m+1}] = 0 \tag{82}$$

$$E[X_1 X_2 \ldots X_{2m+1}] = \sum_{\forall \text{comb. } jkrs\ldots} E[X_j X_k] E[X_r X_s] \tag{83}$$

There are N combinations where $N = \frac{(2m)!}{m! 2^m}$

Example:

$$E[x^{10}] = \frac{(2 \cdot 10)!}{10! 2^{10}} \sigma_x^{10} \tag{84}$$

4 Fourier Series and Fourier Transformations

Fourier transformations as applied to random process theory.

Periodic funciton: $f(t) = f(t + \mathbf{T})$

Fourier Series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(\frac{2\pi nt}{\mathbf{T}}) + b_n \sin(\frac{2\pi nt}{\mathbf{T}}) \right] \tag{85}$$

$$a_n = \frac{2}{\mathbf{T}} \int_{-\frac{\mathbf{T}}{2}}^{\frac{\mathbf{T}}{2}} f(t) \cos(\frac{2\pi nt}{\mathbf{T}}) \mathrm{d}t \quad \text{and} \quad b_n = \frac{2}{\mathbf{T}} \int_{-\frac{\mathbf{T}}{2}}^{\frac{\mathbf{T}}{2}} f(t) \sin(\frac{2\pi nt}{\mathbf{T}}) \mathrm{d}t \tag{86}$$

Exponential Form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/\mathbf{T}} \quad \text{and} \quad c_n = \frac{1}{\mathbf{T}} \int_{-\frac{\mathbf{T}}{2}}^{\frac{\mathbf{T}}{2}} f(t) e^{-i2\pi nt/\mathbf{T}} \mathrm{d}t \tag{87}$$

Fourier Transform As $\mathbf{T} \to \infty$ the Fourier Series degenerates into the Fourier Transform.

$$f(t) = \int_{-\infty}^{\infty} F(w) e^{iwt} \mathrm{d}t \quad \text{and} \quad F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-iwt} \mathrm{d}t \tag{88}$$

Piecewise Continuous For a fourier transform to exist a function must be integrable

$$\int_{-\infty}^{\infty} |f(t)| \mathrm{d}t < \infty \tag{89}$$

and also, the discontinuties must be finite

$$f(t) = \frac{1}{2}[f(t^+) + f(t^-)] \tag{90}$$

5 Random Processes in Frequency Domain

Analysis of stationary random processes can be conducted in the frequency domain.

5.1 Spectral Density Function

Given a stationary random process $X(t)$ the condition $\int_{-\infty}^{\infty} |x(t)| \mathrm{d}t < \infty$ is not satisfied. Thus the fourier transform of the process does not exist. Nevertheless the transform of the correlation function is very usefull.

Define the non-existant fourier transform of X as

$$X(w, \mathbf{T}) = \frac{1}{2\pi} \int_{-\mathbf{T}}^{\mathbf{T}} x(t) e^{-iwt} \mathrm{d}t \tag{91}$$

with

$$E[X(w, \mathbf{t}) X^*(w, \mathbf{t})] = \frac{1}{2\pi}^2 \int_{-\mathbf{T}}^{\mathbf{T}} \int_{-\mathbf{T}}^{\mathbf{T}} R_x(t_1 - t_2) e^{-iw(t_1 - t_2)} \mathrm{d}t_1 \mathrm{d}t_2 \tag{92}$$

And

$$E[X(t_1)X^*(t_2)] = R_x(t_1 - t_2) = R_x(\tau) \tag{93}$$

Now, the limits t_1 and t_2 are related to the period $\mathbf{T}$. By shifting the period and the area of integration cleverly the result becomes

$$E[|X(w, \mathbf{T})|^2] = \frac{1}{(2\pi)^2} \int_{-2\mathbf{T}}^{2\mathbf{T}} (2\mathbf{T} - |\tau|) R_x(\tau) \mathrm{e}^{-iw\tau} \mathrm{d}\tau \tag{94}$$

Taking the limit as $\mathbf{T} \to \infty$:

$$\lim_{\mathbf{T}\to\infty} \left\{ E[|X(w, \mathbf{T})|^2] \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) \mathrm{e}^{-iw\tau} \mathrm{d}\tau \tag{95}$$

Spectral Density: is defined as this limit.

$$S_{XX}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_X(\tau) \mathrm{e}^{-iw\tau} \mathrm{d}\tau \tag{96}$$

and

$$R_X(\tau) = \int_{-\infty}^{\infty} S_{XX}(w) \mathrm{e}^{iw\tau} \mathrm{d}w \tag{97}$$

Furthermore at $\tau = 0$: $R_X(0) = E[X^2] = \sigma_X^2$

A Spectral density function is both real and even $S_X(w) = S_X(-w)$, $S_{XX} = \frac{1}{\pi} \int_{-\infty}^{\infty} R_X(\tau) \cos w\tau \mathrm{d}\tau$. Thus $R_X(\tau) = 2 \int_0^{\infty} S_{XX}(w) \cos w\tau \mathrm{d}\tau$, it is non-negative, and it must decay faster than $\frac{1}{w}$ to be meaningful for a physical process.

The units of spectral density are in general the quantity squared over radians per second.

Narrow Band: a process whose spectral density function is narrow, encompassing a small finite set of frequencies which are adjacent.

Wide Band: a process whose spectral density funciton is wide, encompassing a large finite set of frequencies which are adjacent.

Idealized Gaussian White Noise: A process which encompasses all frequencies – the widest band possible. $S_{XX} = s_0$

Truncated white noize: Bands of frequency are admitted.

Two sided:

$$S_{XX}(w) = \begin{cases} s_0 & w_1 < w < w_2 \\ s_0 & -w_2 < w < w_1 \\ 0 & \text{else} \end{cases} \tag{98}$$

One Sided.

$$S_{XX}(w) = \begin{cases} 2s_0 & w_1 < w < w_2 \\ 0 & \text{else} \end{cases} \tag{99}$$

Example: Idealized Gaussian White Noise

$$R_{xx}(\tau) = 2\pi s_0 \delta(\tau) \tag{100}$$

$$S_{xx}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)\mathrm{e}^{-iw\tau}\mathrm{d}\tau = S_0\int_{-\infty}^{\infty} e^{-iw\tau}\delta(\tau)\mathrm{d}\tau = S_0 \tag{101}$$

The mean square does not exist

$$E[X^2] = \int_{-\infty}^{\infty} S_{xx}(w)\mathrm{d}w = S_0\int_{-\infty}^{\infty}\mathrm{d}w \to \infty \tag{102}$$

Example: Band Limited (Truncated) White Noise

$$S_{xx}(w) = \begin{cases} S_0 & |w| \le w_c \\ 0 & \text{else} \end{cases} \tag{103}$$

The covariance is real

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(w)e^{iw\tau}\mathrm{d}w = \int_{-w_c}^{w_c}\cos(w\tau)\mathrm{d}w = 2S_0\frac{\sin w_c\tau}{\tau} \tag{104}$$

$$E[x^2] = \int_{-w_c}^{w_c} S_0\mathrm{d}w = 2S_0w_c \quad R_{xx}(0) = \lim_{\tau\to 0}\frac{2S_0\frac{\partial}{\partial\tau}\sin w_c\tau}{\frac{\partial}{\partial\tau}\tau} = 2S_0w_c \tag{105}$$

$$R_{xx}(0) = E[x^2] \tag{106}$$

Example: Covariance to Spectral Density

$$R_{xx}(\tau) = A\mathrm{e}^{-\alpha|\tau|}\cos\omega\tau \tag{107}$$

$$E[x^2] = R_{xx}(0) = A \tag{108}$$

$$\begin{aligned} S_{xx} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)\mathrm{e}^{-iw\tau}\mathrm{d}\tau \\ &= \frac{A\alpha}{\pi}\left[\frac{w^2+\alpha^2+\omega^2}{w^4+2w^2(\alpha^2-\omega^2)+(\alpha^2+\omega^2)^2}\right] \end{aligned} \tag{109}$$

5.2 Cross Spectral Density

$$\begin{aligned} S_{XY}(w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)\mathrm{e}^{iw\tau}\mathrm{d}\tau \\ R_{XY}(\tau) &= \int_{-\infty}^{\infty} S_{XY}(w)\mathrm{e}^{iw\tau}\mathrm{d}\tau \end{aligned} \tag{110}$$

Properties of cross correlation:

- Hermitian: $S_{XY}(w) = S^*_{YX}(w) = S_{YX}(-w)$

- If $|R_{XY}(\tau)| < \infty$ then $S_{XY}(w)$ decreases faster than $\frac{1}{w}$ as $w \to \infty$

Coherance Function:

$$\gamma^2_{XY}(w) = \frac{S_{XY}(w)S^*_{XY}(w)}{S_x(w)S_y(w)} = \frac{|S_{XY}(w)|^2}{S_x(w)S_y(w)} \tag{111}$$

By definition $0 \leq \gamma^2 \leq 1$ where 0 is uncorrelated and 1 is fully correlated.

Attenuation and Delay: let $X(t)$ be delayed by α an dattenuated by a

$$Y(t) = aX(t-\alpha) \quad S_{YY}(w) = a^2 S_{XX}(w) \quad S_{XY}(w) = aS_{XX}(w)\mathrm{e}^{iw\alpha} \tag{112}$$

Sum of Processes:

$$Z(t) = X(t) + Y(t) \tag{113}$$

$$S_{ZZ}(w) = S_{XX}(w) + S_{YY}(w) + S_{XY}(w) + S_{YX}(w) \tag{114}$$

$$E[z(t)z(\tau)] = E[(X+Y)(X+Y)] = E[X^2] + E[Y^2] + E[XY] + E[YX] \tag{115}$$

$$S_{ZX}(w) = S_{XX}(w) + S_{YX}(w) \tag{116}$$

Product of Processes:

$$E[ZX] = E[(X+Y)X] = E[X^2] + E[XY] \tag{117}$$

$$S_{ZZ}(w) = S_{XX}(w) + S_{YY}(w)S_{ZX}w = S_{XX}(w) \tag{118}$$

5.3 Derivatives of Stationary Processes

Consider a random process $X(t)$ find $\dot{X}(t)$ and $\ddot{X}(t)$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}R_X(\tau) &= R_{X\dot{X}}(\tau) = R'_X(\tau) \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}R_X(\tau) &= R_{X\ddot{X}}(\tau) = R''_X(\tau) \quad \text{Cross Correlation} \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}R_X(\tau) &= -R_{\dot{X}}(\tau) \qquad \text{Auto Correlation} \\
\frac{\mathrm{d}^3}{\mathrm{d}\tau^3}R_X(\tau) &= R_{X\dot{\ddot{X}}}(\tau) = R'''(\tau) = -R_{\dot{X}\ddot{X}}(\tau) \\
\frac{\mathrm{d}^4}{\mathrm{d}\tau^4}R_X(\tau) &= R_{\ddot{X}\ddot{X}}(\tau)
\end{aligned} \tag{119}$$

Using Spectral density:

$$\begin{aligned}
R_X(\tau) &= \int_{-\infty}^{\infty} S_X(w)\mathrm{e}^{-iw\tau}\mathrm{d}w \\
R''_X(\tau) &= -\int_{-\infty}^{\infty} w^2 S_x(w)\mathrm{e}^{-iw\tau}\mathrm{d}w
\end{aligned}$$

$$
\begin{aligned}
&= -R_{\dot{x}}(\tau) = -\left[\int_{-\infty}^{\infty} w^2 S_x(w) \mathrm{e}^{-iw\tau} \mathrm{d}w\right] \\
S_{\dot{X}}(w) &= w^2 S_X(w) \\
\sigma_{\dot{X}}^2 &= \int_{-\infty}^{\infty} S_{\dot{X}}(w) \mathrm{d}w = \int_{-\infty}^{\infty} w^2 S_x(w) \mathrm{d}w \\
\sigma_{\dot{X}}^2 &= \int_{-\infty}^{\infty} w^4 S_X(w) \mathrm{d}w \\
S_{\ddot{X}}(w) &= w^4 S_X(w) \\
S_{X\dot{X}} &= \mathbf{F}\{R_{X\dot{X}}(\tau)\} = \mathbf{F}\{\frac{\mathrm{d}}{\mathrm{d}t} R_X(\tau)\} = iwS_X(w) \qquad (120)
\end{aligned}
$$

6 Statistical Properties of Random Processes

As the random inputs are transmitted through a linear structural system or filter the output is a narrow band process characterized by that structures response $Y(t)$ and transformed by the frequency response function $H(w)$.

6.1 Expected Rate of Threshold Crossing

Let $v(t)$ be a continuous random process with zero mean. Where ζ is a threshold level. Postulate that the structure fails if:

- First Excursion Failure: $Y(t)$ reaches ζ or $-\zeta$ for the first time
- Fatigue Failure: Failure si the result of accumulated damage – a fixed total.

For the second failure postulate, fatige, counting hte number of crossings in a given time period becomes important. Define a random number η that characterises the number of times $Y(t)$ crosses a threshold level ζ either from above or below.

Crossing times:

$$\eta(\zeta, t_1, t_2) \qquad (121)$$

Mean:

$$E[\eta(\zeta, 0, t)] \qquad (122)$$

Correlation:

$$E[\eta(\zeta, 0, t_1)\eta(\zeta, 0, t_2)] \qquad (123)$$

Let $z[t] = \mathbf{1}[Y(t) - \zeta]$ where $\mathbf{1}$ is the Heavisid Step function.

$$z(t) = \begin{cases} 1 & Y(t) > zeta \\ \frac{1}{2} & Y(t) = \zeta \\ 0 & Y(t) < \zeta \end{cases} \qquad (124)$$

Using this definition

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{1}[Y(t)-\zeta] = \delta[Y(t)-\zeta]\dot{Y}(t) \tag{125}$$

Notice that $z[t]$ is random.

$$\eta(\zeta, t_1, t_2) = \int_{t_1}^{t_2} |\dot{z}(t)|\mathrm{d}t = \int_{t_1}^{t_2} |\dot{Y}(t)\delta[Y(t)-\zeta]|\mathrm{d}t \tag{126}$$

$$E[\eta(\zeta, t_1, t_2)] = \int_{t_1}^{t_2} E\{|\dot{Y}(t)\delta[Y(t)-\zeta]|\}\mathrm{d}t = \int_{t_1}^{t_2} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{y}|\delta(y-\zeta)f_{Y\dot{Y}}(y,\dot{y})\mathrm{d}y\mathrm{d}\dot{y}\right]\mathrm{d}t \tag{127}$$

$$E[\eta(\zeta, t_1, t_2)] = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} |\dot{y}|f_{y,\dot{y}}(\zeta,\dot{y})\mathrm{d}\dot{y}\mathrm{d}t \tag{128}$$

Determine the rate of treshold crossing per unit time $N(\zeta, t)$:

$$N(\zeta, t) = \int_{-\infty}^{\infty} |\dot{y}|f_{Y\dot{Y}}(\zeta,\dot{y})\mathrm{d}\dot{y} \tag{129}$$

Similarly the correlation function of $N(\zeta, t)$

$$\phi_{NN}(\zeta, t_1, t_2) = E[N(\zeta, t_1)N(\zeta, t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{y}_1||\dot{y}_2|f_{Y_1Y_2}(\zeta,\dot{y}_1,\dot{y}_2)\mathrm{d}\dot{y}_1\mathrm{d}\dot{y}_2 \tag{130}$$

To count only upcrossings of $zeta$

$$E[N_+(\zeta, t)] = \int_0^{\infty} \dot{y}f_{Y\dot{Y}}(\zeta, y)\mathrm{d}\dot{y} \tag{131}$$

For a stationary random process

$$E[N_+(\zeta, t)] = \frac{1}{2}E[N(\zeta, t)] \tag{132}$$

6.1.1 Upcrossings of Gaussian Random Process

$$N_+(\zeta, t) = N_+(\zeta) \quad \text{Statinoary Random Process} \tag{133}$$

$$E[N_+(\zeta)] = \int_0^{\infty} \dot{y}\frac{1}{2\pi\sigma_y\sigma_{\dot{y}}}\mathrm{e}^{-\frac{1}{2}(\frac{\zeta^2}{\sigma_y^2}+\frac{\dot{y}^2}{\sigma_{\dot{y}}^2})}\mathrm{d}\dot{y} = \frac{1}{2\pi\sigma_y\sigma_{\dot{y}}}\mathrm{e}^{\frac{-\zeta^2}{2\sigma_y^2}}\int_0^{\infty} \dot{y}\mathrm{e}^{-\frac{\dot{y}^2}{2\sigma_{\dot{y}}^2}}\mathrm{d}\dot{y} \tag{134}$$

Let $\mu = \frac{\dot{y}}{2\sigma_{\dot{y}}^2}$, and $\mathrm{d}u = \frac{\dot{y}}{\sigma_{\dot{y}}^2}\mathrm{d}y$

$$E[N_+(\zeta)] = \frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_Y}\mathrm{e}^{\frac{-\zeta^2}{2\sigma_y^2}} \tag{135}$$

Since the variances depend on the spectral density. If the spectral density is known the upcrossing rate can be found. The zero upcrossing rate occurs at $\zeta = 0$

$$E[N_+(0)] = \frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_Y} = \frac{1}{2\pi}\left[\frac{\int_{-\infty}^{\infty} w^2 S_{YY}(w)\mathrm{d}w}{\int_{-\infty}^{\infty} S_{YY}(w)\mathrm{d}w}\right] \tag{136}$$

For a narrow band process:

$$E[N_+(0)] = f_e = \frac{1}{2\pi}\sqrt{\frac{2S_0 w_0^2 \Delta w}{2\S_0 \Delta w}} = \frac{w_0}{2\pi} \tag{137}$$

For a gaussian process: as variance increases the change for upcrossings decreases:

$$\frac{N_+(\zeta)}{N_+(0)} = \mathrm{e}^{\frac{-\zeta^2}{2\sigma_y^2}} \tag{138}$$

Example: Non-Gaussian Process Upcrossing Rate Consider Vibrations of a Nonlinear Single-Degree-of-Freedom system – a Duffing oscillator.

$$\ddot{Y} + c\dot{Y} + w_0^2(Y + \epsilon Y^3) = F(t) \tag{139}$$

Using pertubation methods, the solution approaches $f_{Y\dot{Y}}(y, \dot{y})$ for the joint density funciton where:

$$f_{Y\dot{Y}}(y, \dot{y}) = \mathbf{C}\exp\left\{\frac{-c}{\pi S_0\left[\frac{\dot{y}^2}{2} + w_0^2(\frac{y^2}{2} + \epsilon\frac{y^4}{4})\right]}\right\} \tag{140}$$

To find $\mathbf{C}$ normalize the distribution.

$$\mathbf{C} \approx \frac{cw_0}{2\pi\pi S_0}\left(1 - \epsilon\frac{3}{4}\frac{\pi S_0}{cw_0^2}\right)^{-1} \tag{141}$$

Using this the statistical terms can be found:

$$E[N_t(0)] = \frac{w_0}{2\pi}(1 - \epsilon\frac{3}{4}\frac{\pi}{C}\frac{S_0}{w)0^2})^{-1} \tag{142}$$

Notice that the expected value of upcrossings for the nonlnear system is greater than that of the linear system – this is generally the case. Also frequency is now a functiononf the amplitude of the forcing function.

6.2 Distribution of Peaks

Peak: occurs in a sample function $Y(t)$ when $\dot{Y}(t) = 0$ and $\ddot{Y}(t) < 0$

Valley: occurs in a sample function $Y(t)$ when $\dot{Y}(t) = 0$ and $\ddot{Y}(t) > 0$

6.3 The Rate of Peaks

A Peak Rate [Middleton, Rice].

$$P_R(t) = |\ddot{Y}(t)|\delta[\dot{Y}(t)]\mathbf{1}[Y(t) - \zeta]. \tag{143}$$

The number of peaks in time $t \in (t_1, t_2)$

$$\epsilon[\zeta, t_1, t_2] = \int_{t_1}^{t_2} P_R(t)\mathrm{d}t. \tag{144}$$

The expected total number of peaks in a time interval

$$E[\epsilon(\zeta, t_1, t_2)] = \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\ddot{y}|\delta|\dot{y}|\mathbf{1}(y-\zeta) f_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y})\mathrm{d}y\mathrm{d}\dot{y}\mathrm{d}\ddot{y}. \tag{145}$$

Define $M[\zeta, t]$ as the number of peaks above ζ per unit time

$$\epsilon(\zeta, t_1, t_2) = \int_{t_1}^{t_2} M[\zeta, t]\mathrm{d}t. \tag{146}$$

The expected peak rate:

$$\begin{aligned} E[M(\zeta, t)] &= -\int_{\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}\dot{y}\int_{-\infty}^{0} f_{Y\dot{Y}\ddot{Y}}(y, \dot{y}, \ddot{y})\delta\dot{y}\mathbf{1}(y-\zeta)\mathrm{d}\ddot{y} \\ &= -\int_{\zeta}^{\infty}\int_{-\infty}^{0} f_{Y\dot{Y}\ddot{Y}}(y, 0, \ddot{y})\ddot{y}\mathrm{d}y\mathrm{d}\ddot{y}. \end{aligned} \tag{147}$$

To get a total number of points let $\zeta \to \infty$

$$E[M_T(t)] = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y\dot{Y}\ddot{Y}}(y, 0, \ddot{y})\ddot{y}\mathrm{d}y\mathrm{d}\ddot{y}. \tag{148}$$

If $Y(t)$ is a stationary and gausian process:

$$f_{Y\dot{Y}\ddot{Y}}(y, 0, \ddot{y}) = \frac{1}{(2\pi)^{\frac{3}{2}}|\Lambda|}\exp\left\{-\frac{1}{2|\Lambda|}(\sigma_2^2\sigma_3^2 y^2 + 2\sigma_2^4 y\ddot{y} + \sigma_1^2\sigma_2^2\dot{y}^2)\right\}. \tag{149}$$

Here, $\Lambda = \sigma_2^2(\sigma_1^2\sigma_3^2 - \sigma_2^4)$, $\sigma_1^2 = \sigma_y^2 = \int_{-\infty}^{\infty} S_{YY}(w)\mathrm{d}w$, $\sigma_2^2 = \sigma_{\dot{y}}^2 = \int_{-\infty}^{\infty} wS_{YY}(w)\mathrm{d}w$, and $\sigma_3^2 = \sigma_{\ddot{y}}^2\int_{-\infty}^{\infty} w^2 S_{YY}(w)\mathrm{d}w$.

Substitute values and integrate to find:

$$E[M_T(t)] = E[M_T] = \frac{1}{2\pi}\frac{\sigma_3}{\sigma_2} = \frac{1}{2\pi}\frac{\int_{-\infty}^{\infty} w^4 S_{YY}\mathrm{d}w}{\int_{-\infty}^{\infty} w^2 S_{YY}\mathrm{d}w}. \tag{150}$$

6.3.1 The Probability Density of a Peak Distribution

The conditional probability distribution function of peak magnitude at time t is equal or than less than ζ.

$$F_I(\zeta,t)\left[1-\frac{E[M(\zeta,t)]}{E[M_T(t)]}\right] \tag{151}$$

Differentiate to find the probability density function:

$$f_I(\zeta,t)=\frac{\partial}{\partial z}F(\zeta,t)=-\frac{1}{E[M_T(t)]}\frac{\partial}{\partial\zeta}\left\{E[M(\zeta,t)]\right\}. \tag{152}$$

Using the Leibniz rule of differentiation:

$$F_I(\zeta,t)=-\frac{1}{E[M_T(t)]}\int_{-\infty}^{\infty}\ddot{y}f_{Y\dot{Y}\ddot{Y}}(\zeta,0,\ddot{y})\mathrm{d}\dot{y}. \tag{153}$$

For a stationary random process $f_I(\zeta,t)=f_I(\zeta)$. Assume $Y(t)$ is a Gaussian random process substituting equations given and integrating:

$$\begin{aligned} f_I(\zeta) &= \frac{(1-\alpha^2)^{\frac{1}{2}}}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{\zeta^2}{2\sigma_1^2(1-\alpha)^2}\right\} \\ &\quad +\frac{\alpha\zeta}{2\sigma_1}\left\{1+\mathrm{erf}\left[\frac{\zeta}{\sigma_1}(2\alpha^{-2}-2)^{-\frac{1}{2}}\right]\right\}\exp\left(-\frac{\zeta^2}{2\sigma_1^2}\right) \end{aligned} \tag{154}$$

where $\alpha=\frac{E[N_+(0)]}{E[M_T]}=\frac{\sigma_2^2}{\sigma_2\sigma_3}$. α ranges from 0 to 1. If $\alpha=0$ for the most part the process fluctuates above or below zero, since it is stationary some crossings do occur. If $\alpha=1$ the process behaves like a narrow band process, one crossing per fluctuation.

$$\begin{aligned} f_I(\zeta) &\approx \frac{1}{\sqrt{2\pi}\sigma_1}\mathrm{e}^{-\zeta^2 2\sigma_1^2} \quad \text{for} \quad \alpha=0: \quad \text{Gaussian} \\ f_I(\zeta) &= \frac{\zeta}{\sigma_1}\mathrm{e}^{\frac{-\zeta^2}{2\sigma_1^2}} \quad \text{for} \quad \alpha=0: \qquad\qquad \text{Rayleigh} \end{aligned} \tag{155}$$

6.3.2 Usefull Approximations

If reversals can be neglected $E[M(\zeta)]\approx E[N_+(\zeta)]$ and $E[M_T]\approx E[N_+(0)]$.

$$F_I(\zeta)=-\frac{1}{E[N_+(0)]}\frac{\mathrm{d}}{\mathrm{d}\zeta}\int_0^{\infty}\dot{y}f_{Y\dot{Y}}(\zeta,\dot{y})\mathrm{d}y. \tag{156}$$

For a gaussian random process $E[N_+(0)]=\frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_Y}$. Thus, if an upcrossing problem is solved the peak crossing problem is solved automatically. Thus for a gaussian proces sit can be shown quickly that the narrow band peak crossing distribution is a Rayleigh distribution.

Example 1: Obtain the Probability that, for a narrow gaussian process any peak exceeds $1\sigma_Y, 2\sigma_Y, 3\sigma_Y$.
The probability of peak crossings is the related to the Rayleigh distribution:

$$\int_{\zeta_1}^{\infty} f_I(\zeta)\mathrm{d}\zeta = \int_{\zeta_1}^{\infty} \frac{\zeta}{\sigma_Y^2} \mathrm{e}^{\frac{-\zeta^2}{2\sigma_Y^2}} \mathrm{d}\zeta = \mathrm{e}^{\frac{-\zeta_1^2}{2\sigma_Y^2}} \tag{157}$$

Example 2: Calculate the zero upcrossing rate and total number of peaks per unit time if the response spectral constant s_0 between w_1, w_2 and $-w_1, -w_2$.

$$E[N_+(0)] = \frac{1}{2\pi}\frac{\sigma_{\dot{Y}}}{\sigma_Y} = \frac{1}{2\pi}\sqrt{\frac{(w_2^3 - w_1^3)}{3(w_2 - w_1)}} \tag{158}$$

$$E[M_T] = \frac{1}{2\pi}\frac{\sigma_{\ddot{Y}}}{\sigma_{\dot{Y}}} = \frac{1}{2\pi}\sqrt{\frac{3(w_2^5 - w_1^5)}{5(w_2^3 - w_1^3)}} \tag{159}$$