# Symbolic method for defining test functions which satisfy ellipticity, boundary conditions lower order field behaviors exactly

Elisabeth Malsch

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In **1877** Lord Rayleigh developed the approximation method for problems of vibration.



In **1908** Walter Ritz independently presented the general approximation method.



$$\mathcal{L}[\vec{u}(x,y)] \approx [\mathsf{A}]^{n\times n}\{\vec{u}_n\} = \{\vec{f}_n\}$$

Construct an approximate solution to any **elliptic partial differential equation**

construct variational form

↓

assume a solution

↓

minimize

The **test function** is a particular assumed solution which satisfies the following properties:

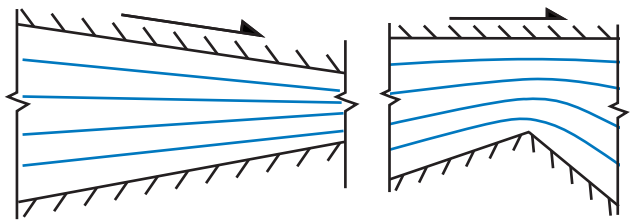
- Smoothness
- Essential boundary conditions
- Linear independence

# Solution to a homogenous elliptic problem

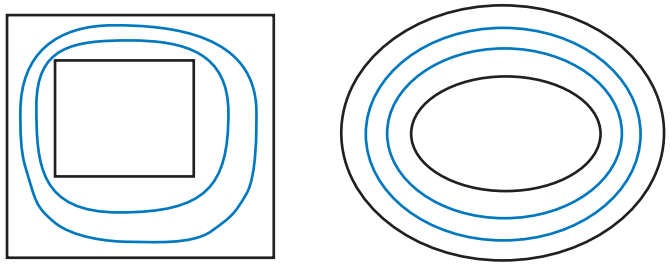
Maximum and minimum values *must* occur on the boundary.

Functions *must* be smooth in the interior.

**Examples:**



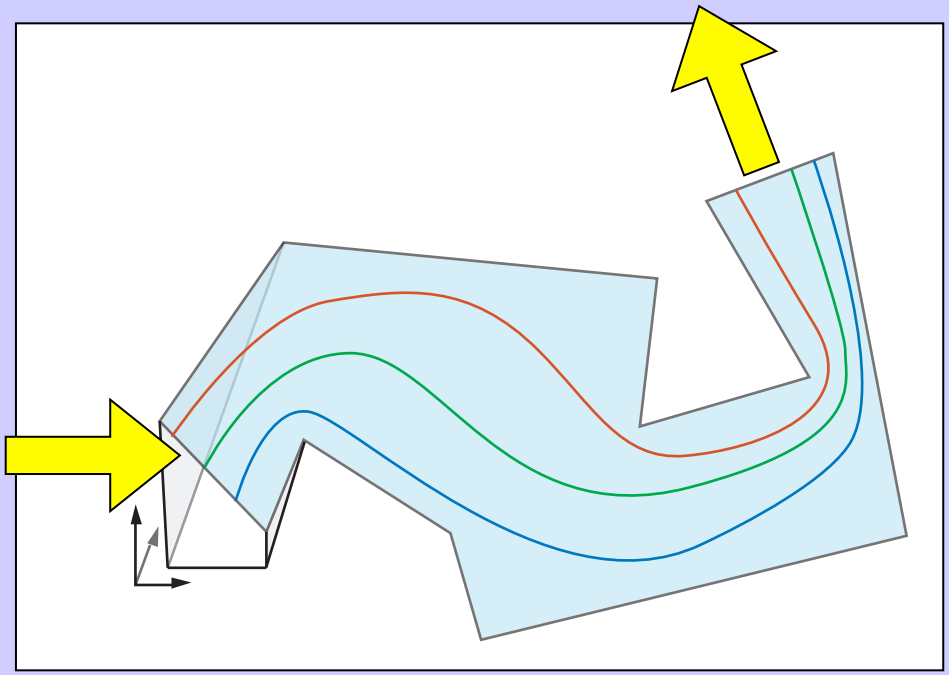
• Irrotational fluid flow



• Torsion stress function/ Membrane

• Temperature distribution ...

$$\nabla^2 u(x, y) = 0$$



**Racetrack analogy**

***an elliptic race track would:***

Never let a runner run uphill

Lanes only bend smoothly

Lanes do not cross

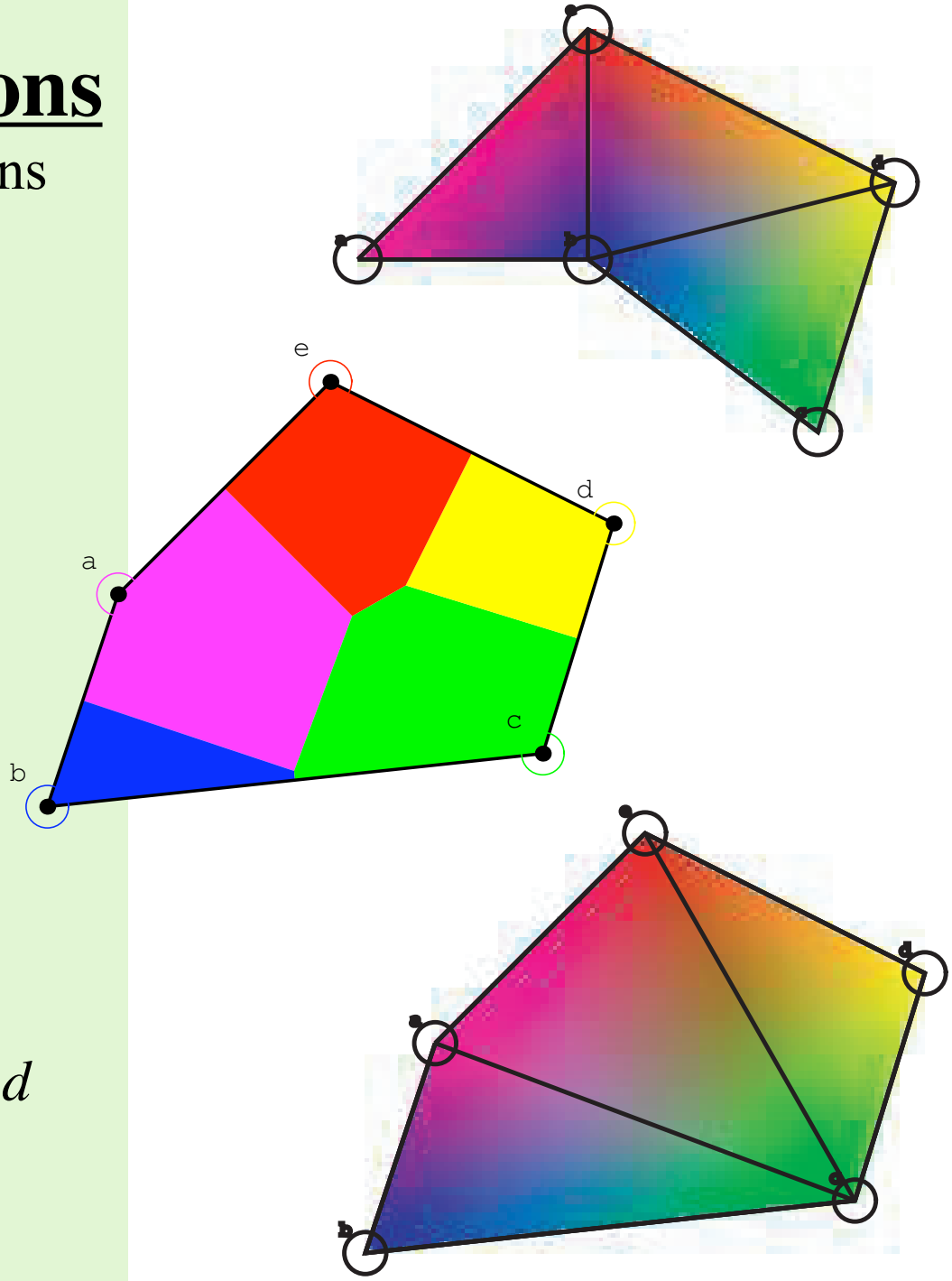
# Available 2-D test functions

- Some simple two dimensional domains
  - Courant's triangle
  - Taig's quadrilateral
- Wachspress convex element (1975)

# Alternative approaches

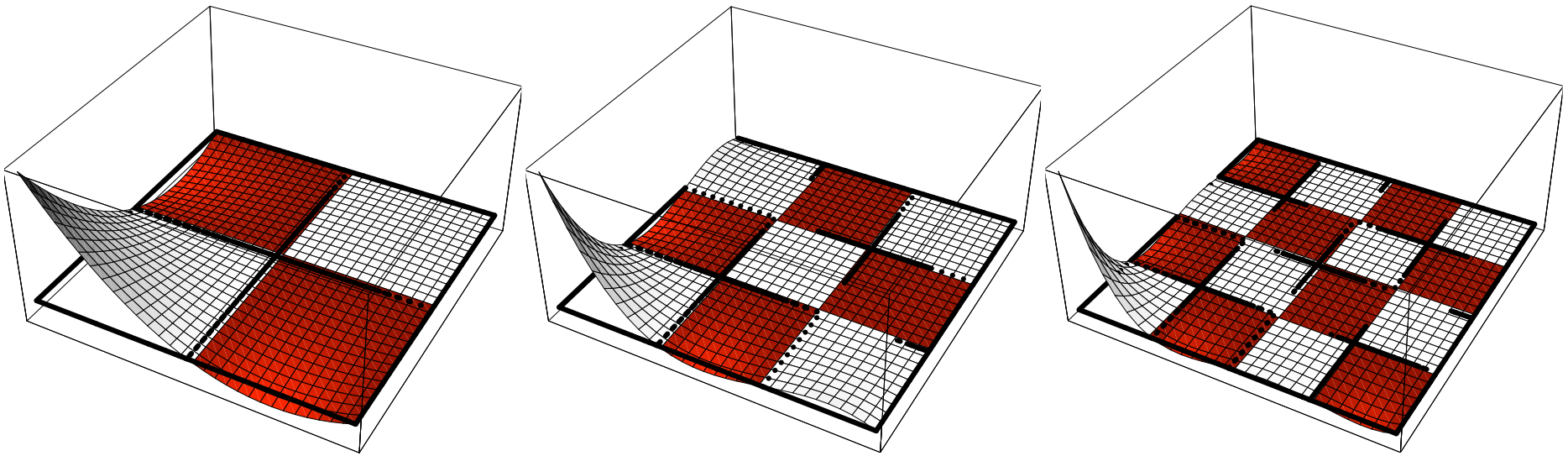
- *Finite Element Method*
  - sacrifice smoothness
- *Boundary Element Method*
  - sacrifice boundary conditions
  - solve the field equations

*The test function conditions are satisfied only in an average sense.*



# Counter Example:

Higher order Lagrange elements



**Unrealistic representations:** Negative Kelvin

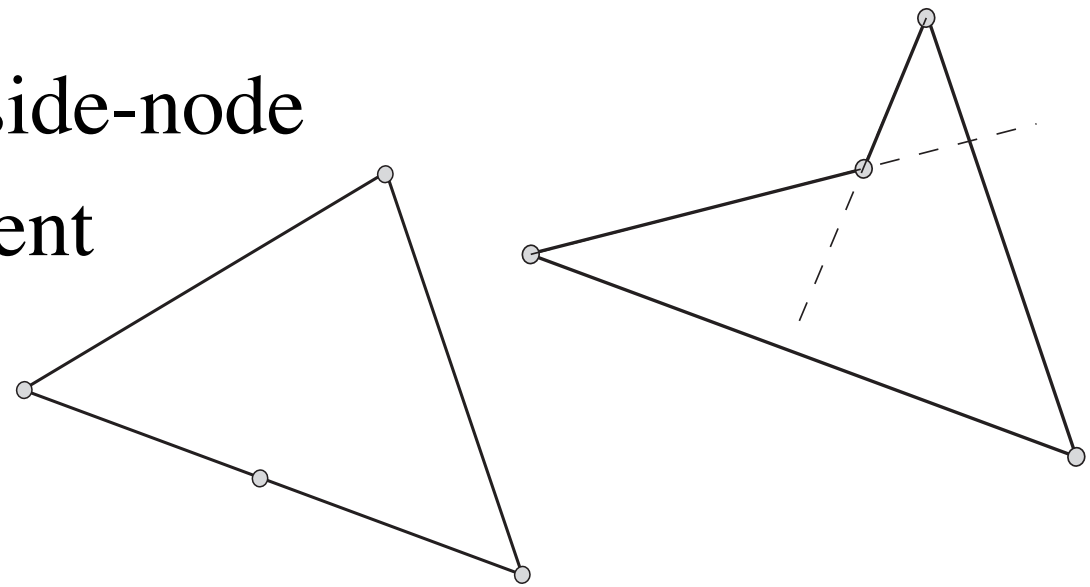
**Inability to reproduce constant and linear fields:**

rigid body motion produces strain energy

**Non-uniform convergence:** instability

# Unavailable test function formulations

- Linear edged side-node
- Concave element
- Inclusions

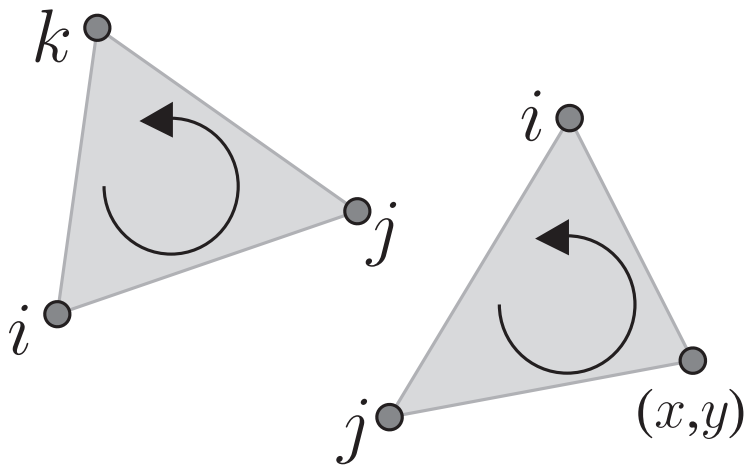


Polynomial and rational polynomial formulations are insufficient

$$\frac{a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy}{b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 y^2 + a_5 xy}$$

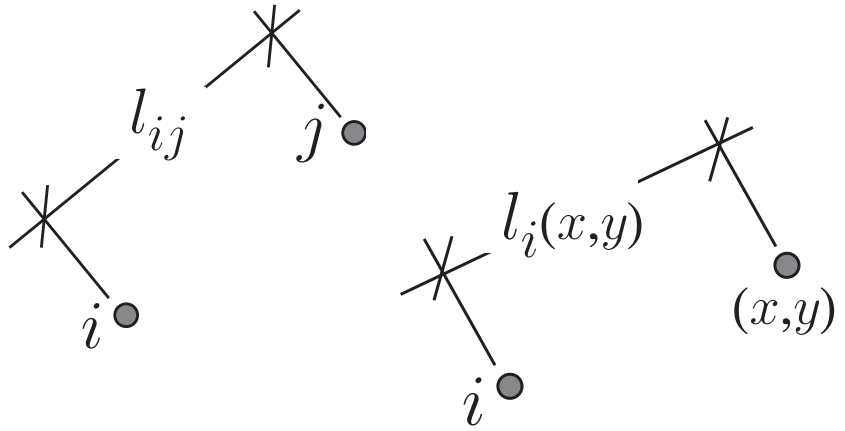
Other methods, such as conformal mapping apply only to specific shapes and result in transcendental functions, not in algebraic functions.

# Area

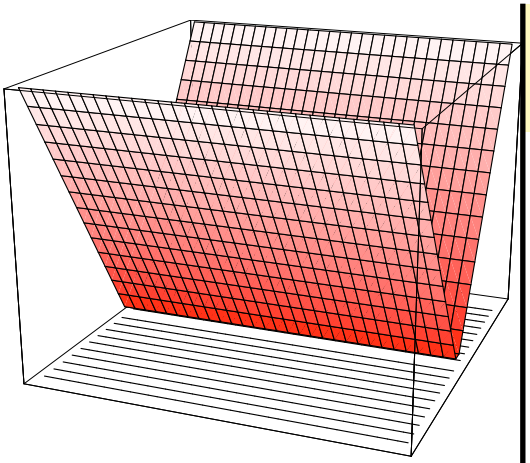



$$\triangle(i,j,k) = \frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}$$

# Length

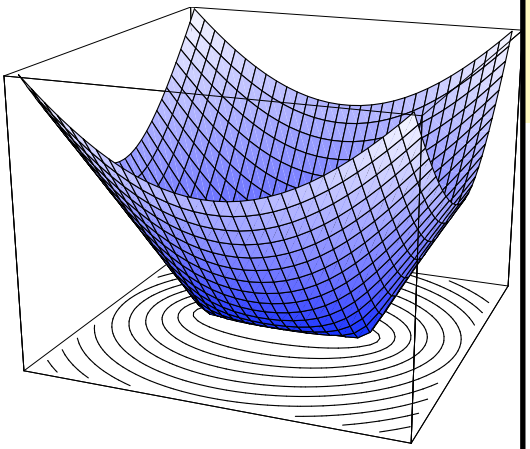



$$l_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$



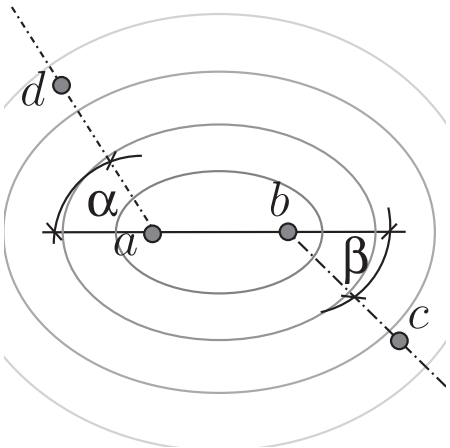
$$\triangle(\vec{x}, i, j)$$

The linear function which is zero along the line connecting points **i** and **j** can be described as the area of the triangle defined by the points **i**, **j** and the variable point **(x,y)**.



$$r_{ij}(\vec{x}) = l_i(\vec{x}) + l_j(\vec{x}) - l_{ij}$$

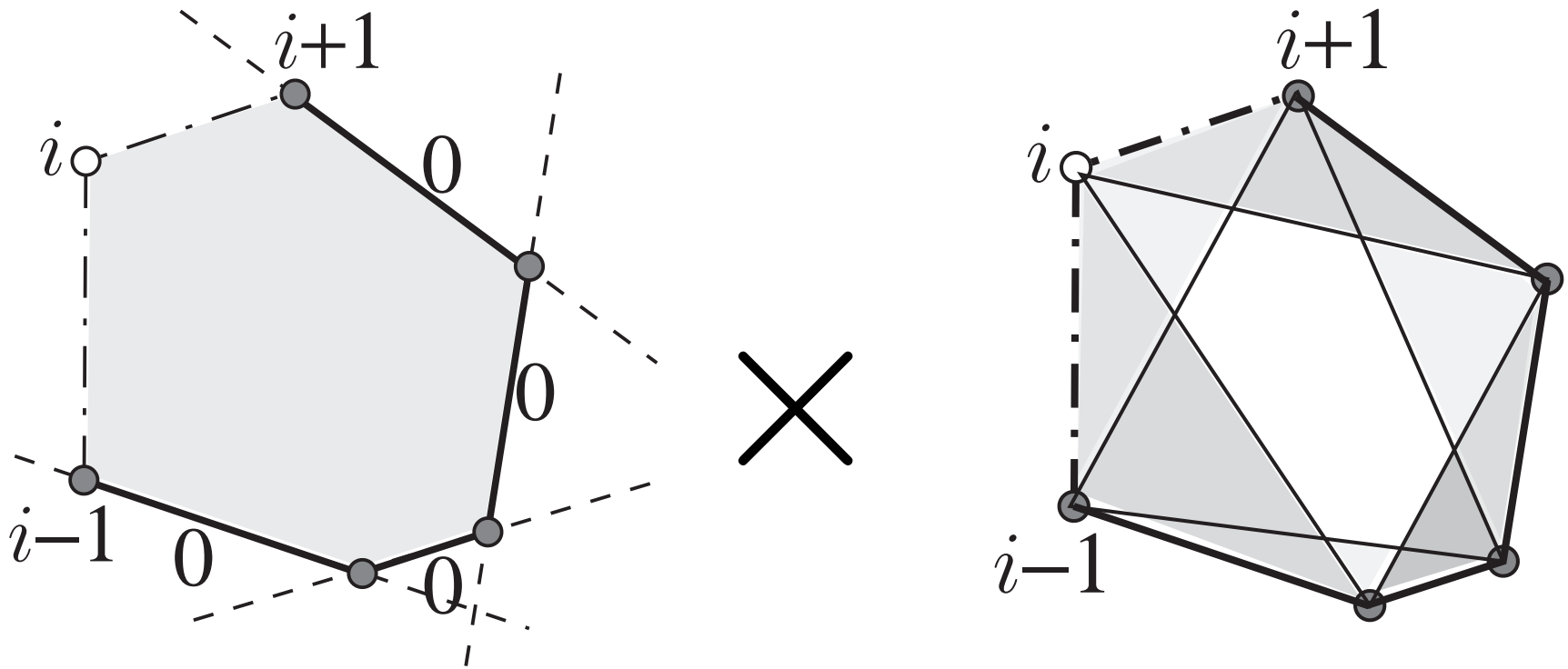
This elliptical function constructed as a combination of line lengths is zero only along the finite line connecting point **i** and **j**.




$$r_{dabc}(\vec{x})$$

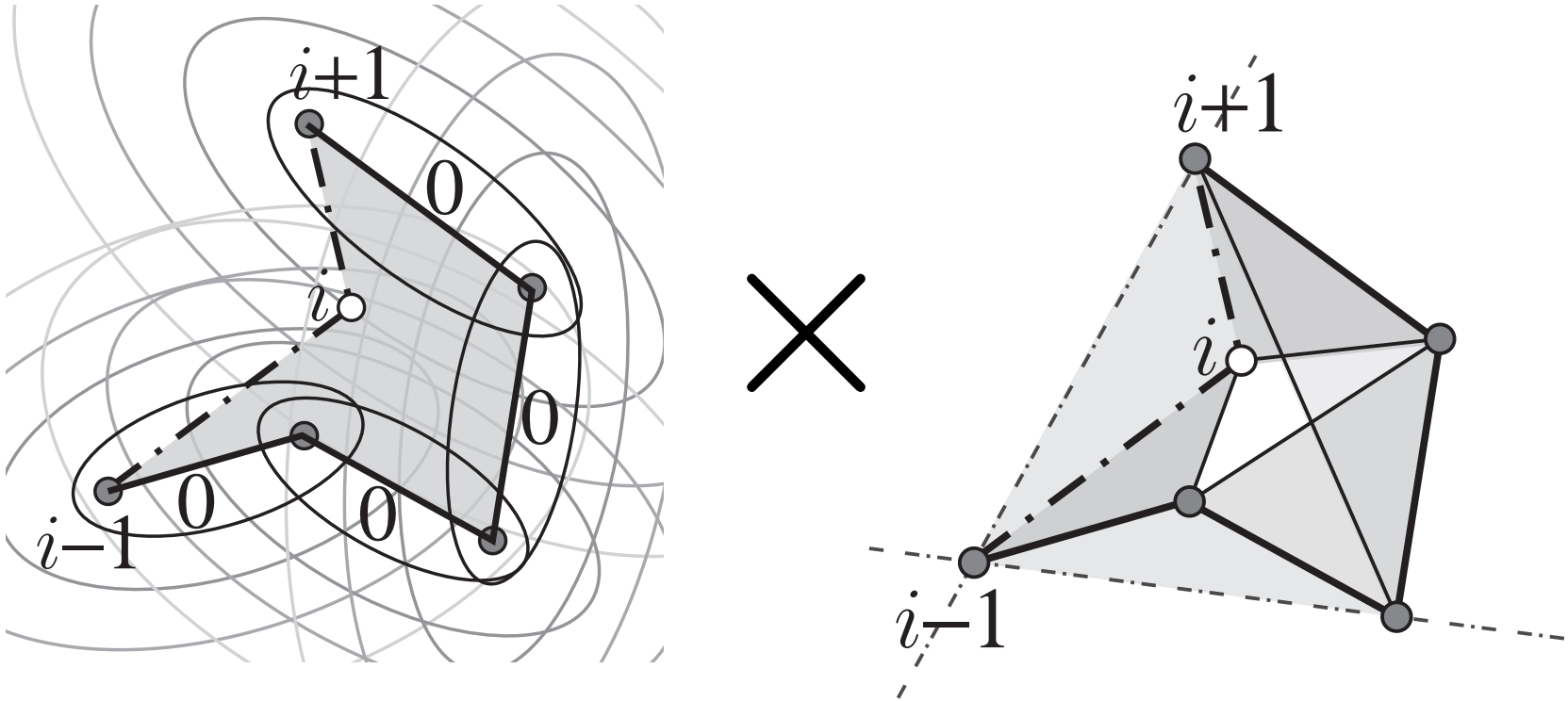
A elliptical function which is linear along the lines **da** and **bc** and zero and minimum only along the line **ab** can be constructed as a combination of line lengths and areas.

# Polynomial numerator




$$\prod_{\substack{i \neq j \\ i \neq k}} \triangle(\vec{x}, j, k) \qquad \triangle(i-1, i, i+1)$$

# Irrational polynomial numerator



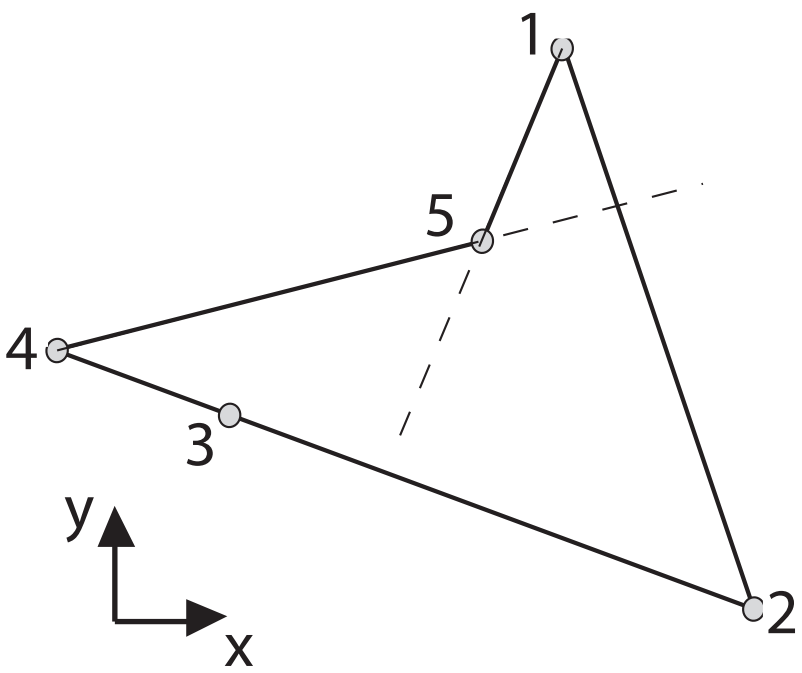

$$\prod_{\substack{i \neq j \\ i \neq k}} r_{jk}(\vec{x}) \qquad |\triangle(i-1, i, i+1)|$$

**Final step:** normalization

$$N_i(\vec{x}) = \frac{k_i s_i(\vec{x})}{\sum_{i=j}^{n} k_j s_j(\vec{x})}$$

- The numerator is zero at all points where the function $N_i(\vec{x})$ has no influence.
- The denominator is constant for the triangle and rectangle.

# Example:




**Numerators:**

1) $k_1\, r_{23}(x,y) r_{34}(x,y) r_{45}(x,y)$

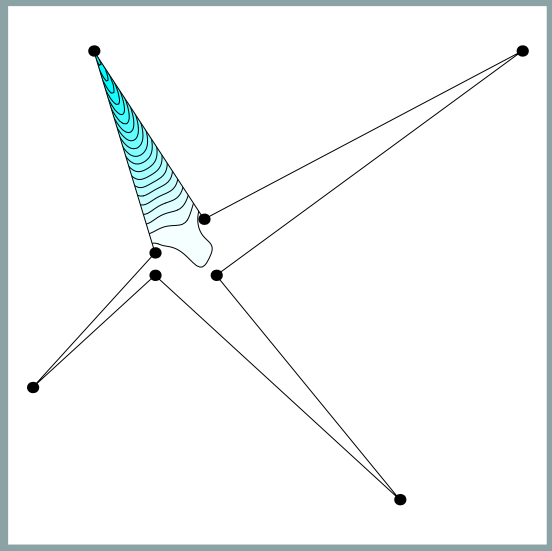
2) $k_2\, r_{34}(x,y) r_{45}(x,y) r_{51}(x,y)$

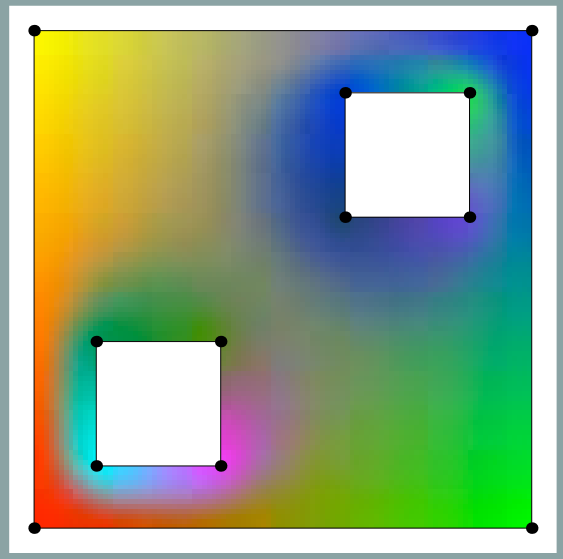
3) $k_3\, r_{45}(x,y) r_{51}(x,y) r_{12}(x,y)$

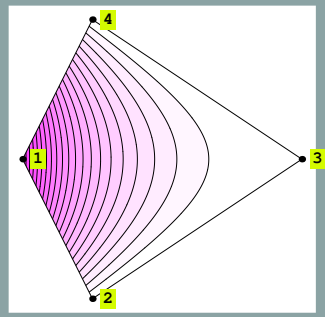
4) $k_4\, r_{51}(x,y) r_{12}(x,y) r_{23}(x,y)$

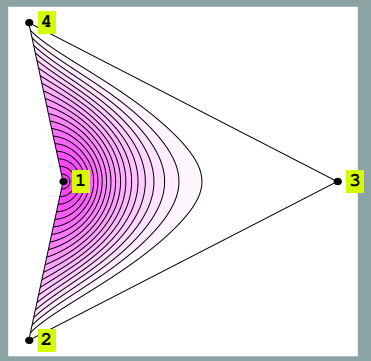
5) $k_5\, r_{12}(x,y) r_{23}(x,y) r_{34}(x,y)$

$$N_1(x,y) = \frac{k_1 r_{23}(x,y) r_{34}(x,y) r_{45}(x,y)}{k_1 r_{23}(x,y) r_{34}(x,y) r_{45}(x,y) + \cdots + k_5 r_{12}(x,y) r_{23}(x,y) r_{34}(x,y)}$$



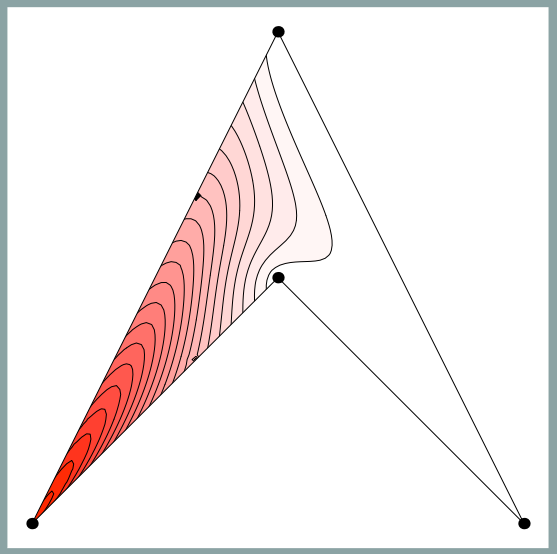


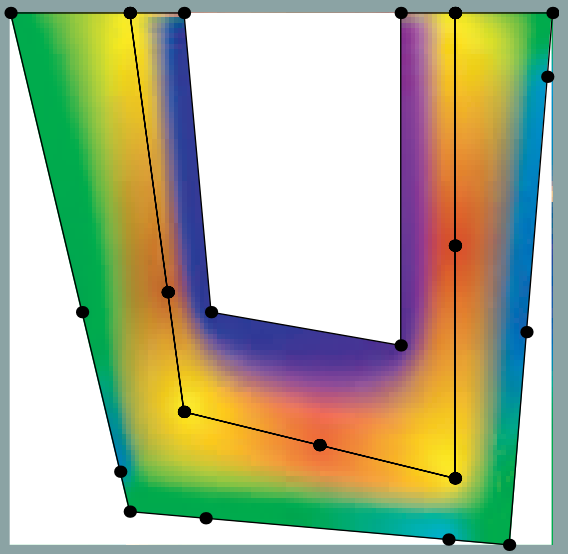






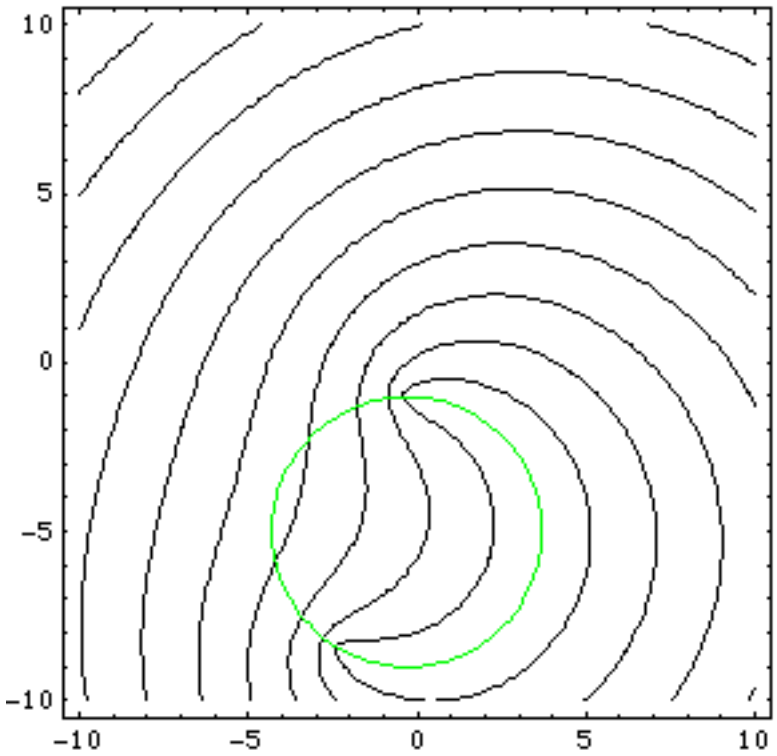



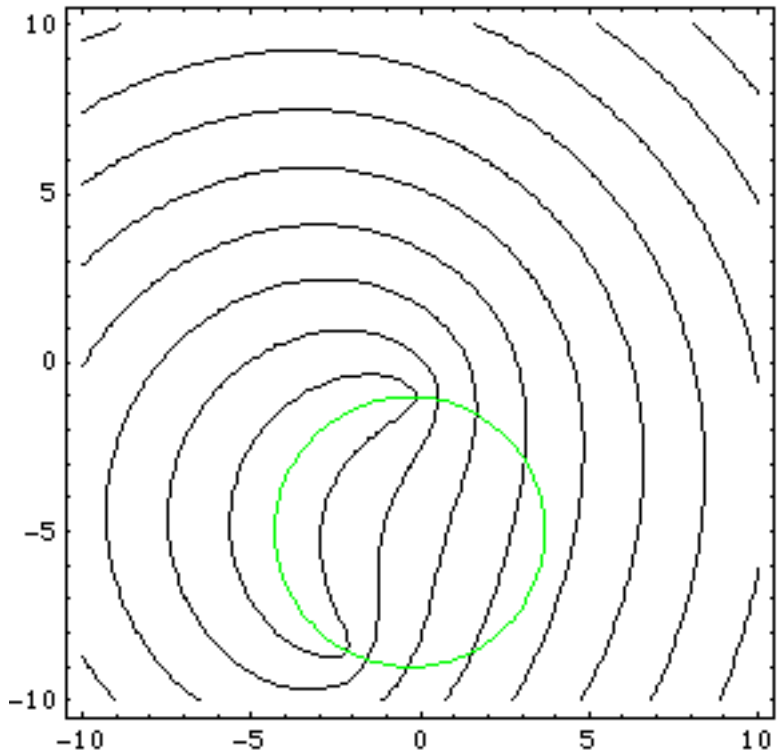


# Any polygonal shape can be described

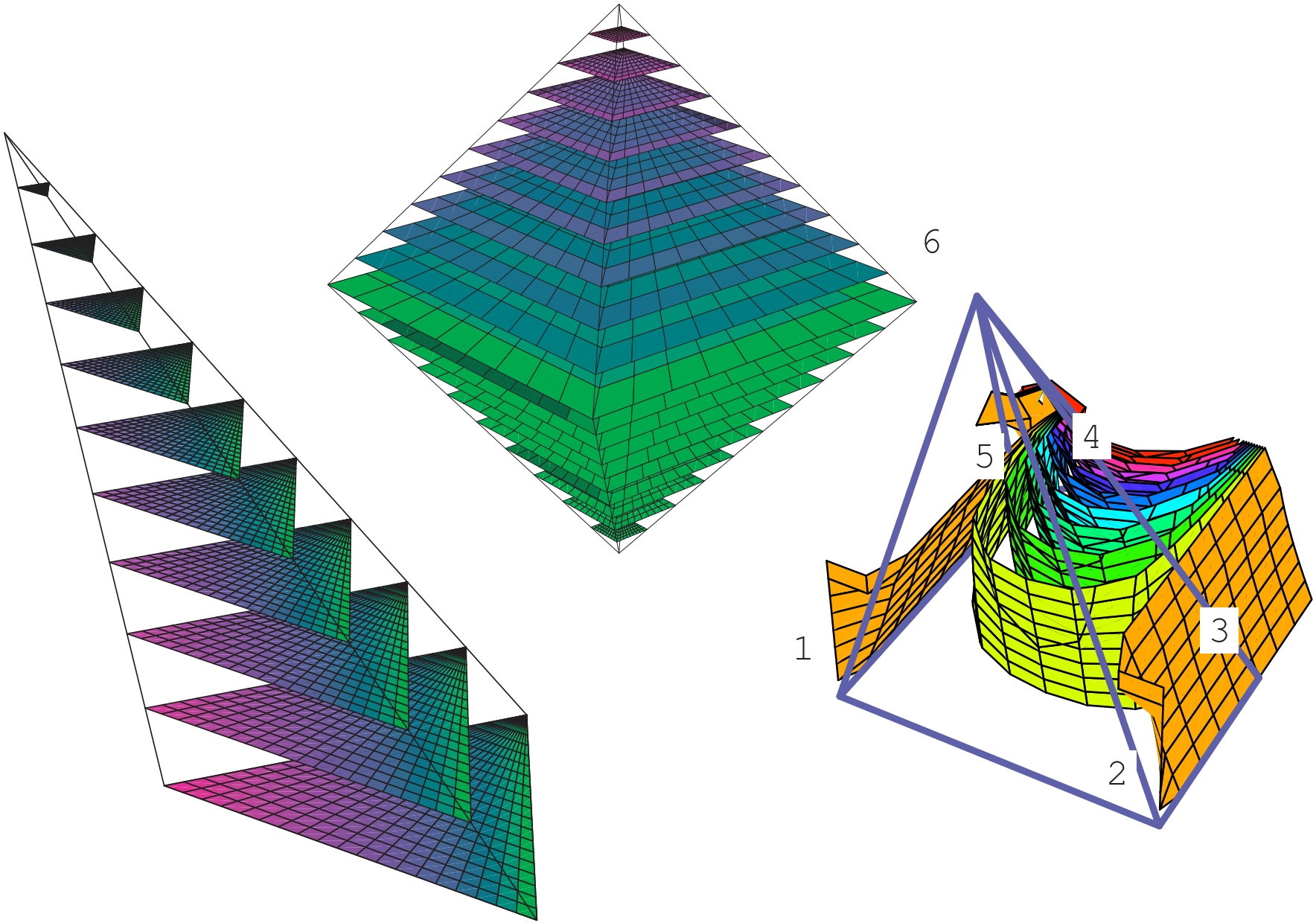
| *geometric test function* | *finite element* | *boundary element* |
|---|---|---|
| no mesh | mesh | no mesh |
| general | general | specific fundamental solution |
| rational & irrational polynomial | polynomial & parametric | algebraic & trancendental |
| bounded | unbounded | unbounded |
| not closed* | closed | closed |
| stable | unstable | stable |

# Boundary with radius of curvature




$$\sqrt{l^2_{(ab)}(\Delta(\vec{x},a,b) - \Delta(O,a,b) + R\, l_{(ab)} l_a(\vec{x}) l_b(\vec{x}) + \Delta(O,a,b)(l_a(\vec{x})^2 + l_b(\vec{x})^2)}$$

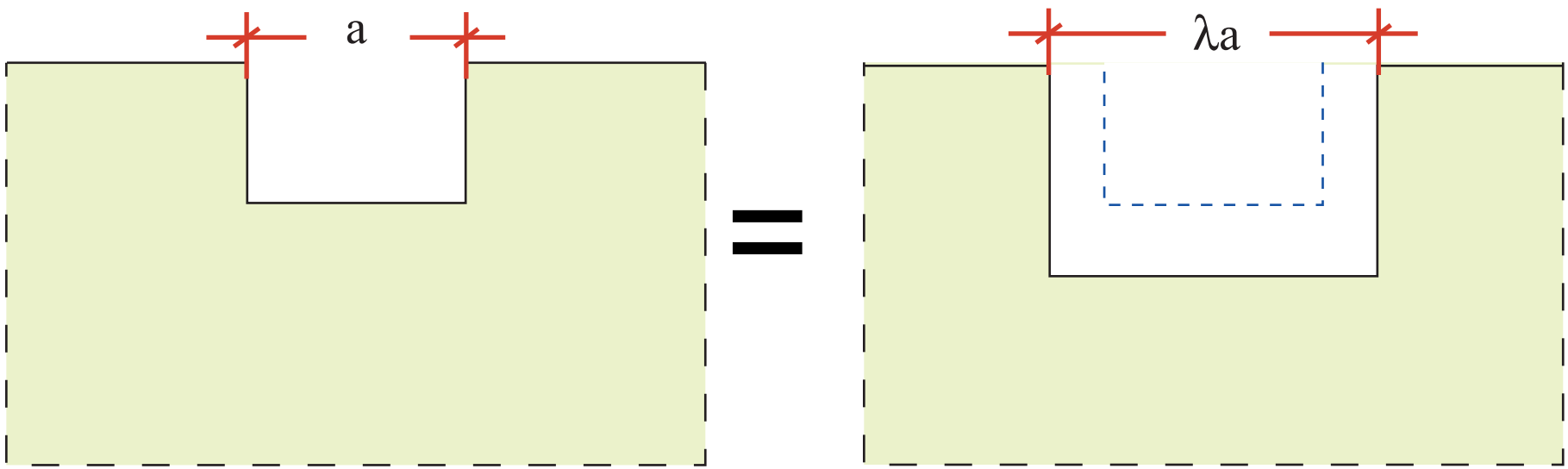



# Three dimensional approaches

# Infinite Domain: Cloning

Simplest example: the stiffness matrix

- *Similar shapes of the same materials have equal stiffness.*




- *The stiffness of the difference (the concave cell) can determine the stiffness of the whole unbounded domain.*

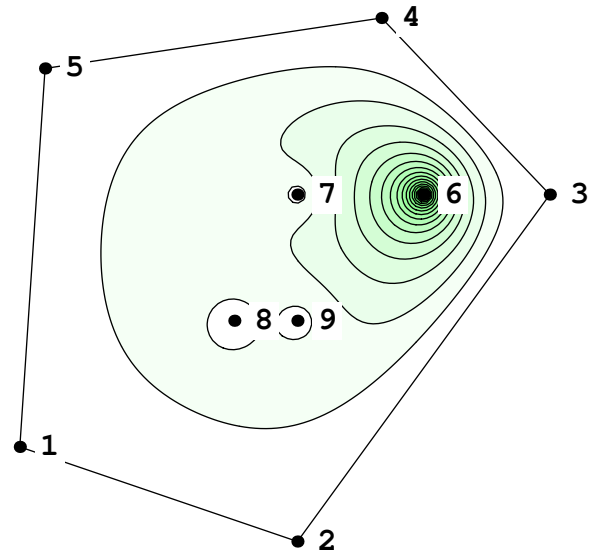
# Further study

- **Adding interior point sources**

  *Ensuring closure*

- **Integration of energy densities**

  *Algebraic and numerical methods for rational polynomial and irrational polynomials*




- **Curved boundaries, higher dimensions**

  *Methods used to extend the convex element formulation can be applied to the developed general method.*

- **Satisfying additional constraints**

  *For example, incompressibility & the plane section assumption.*

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