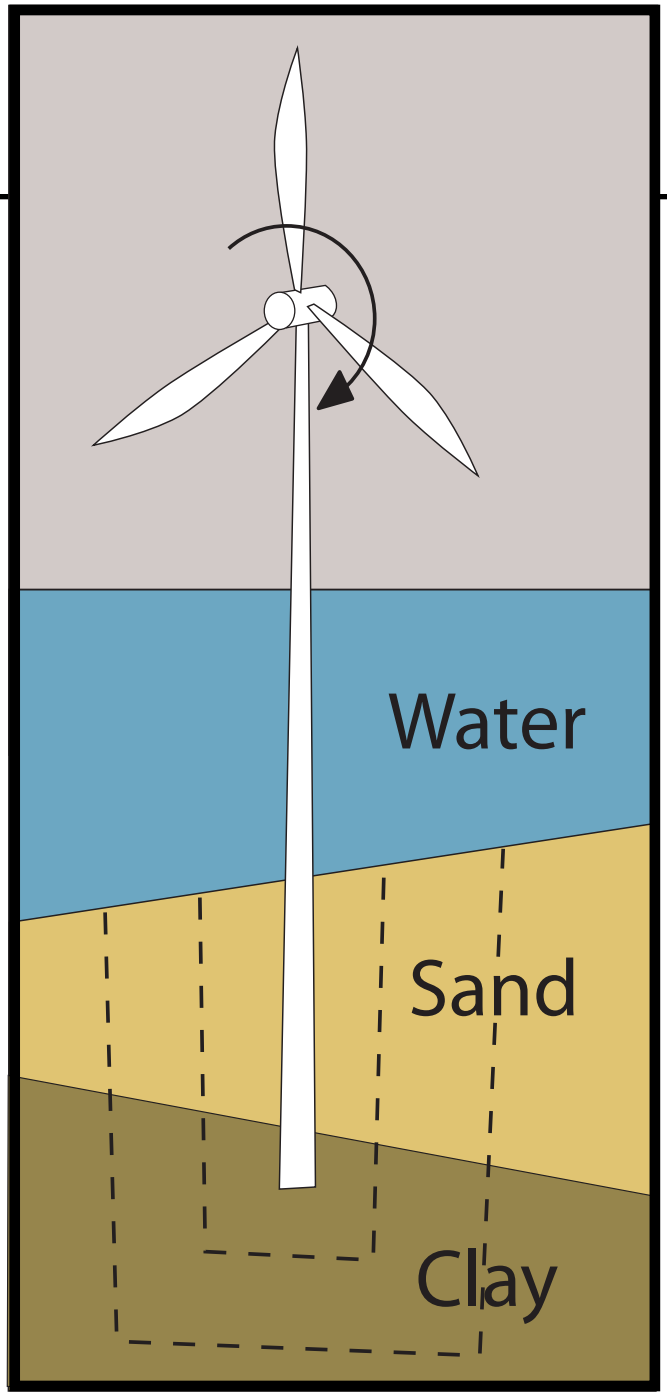


Institut für Angewandte Mechanik
Technische Universität Braunschweig

Coupling finite and semi-finite domains

to model soil-structure interaction

E. A. Malsch and L. Lehmann

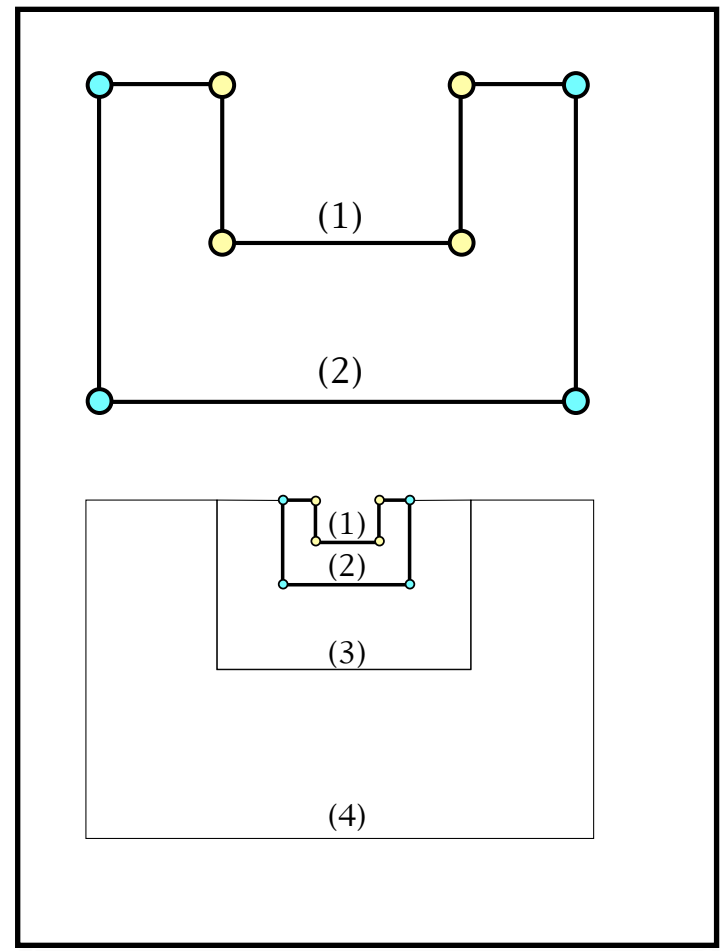
ECCOMAS: Coupled Problems
May 25, 2005 -- Santorini Island, Greece

Proportional Partitioning

- Tile a domain with elements which have similar shapes
- On a reasonably homogeneous domain the stiffness of one cell prescribes the stiffness of the whole domain

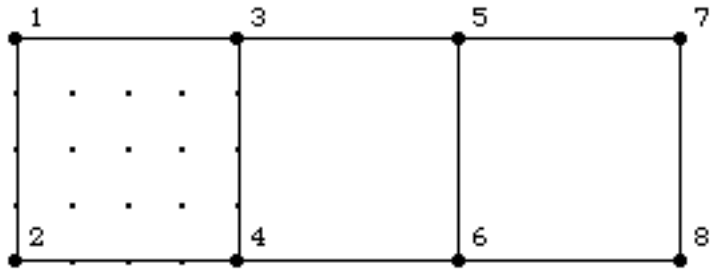
$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} \{u_1\} \\ \{u_2\} \end{array}\right\} = \left\{\begin{array}{c} \{f_1\} \\ \{f_2\} \end{array}\right\}$$

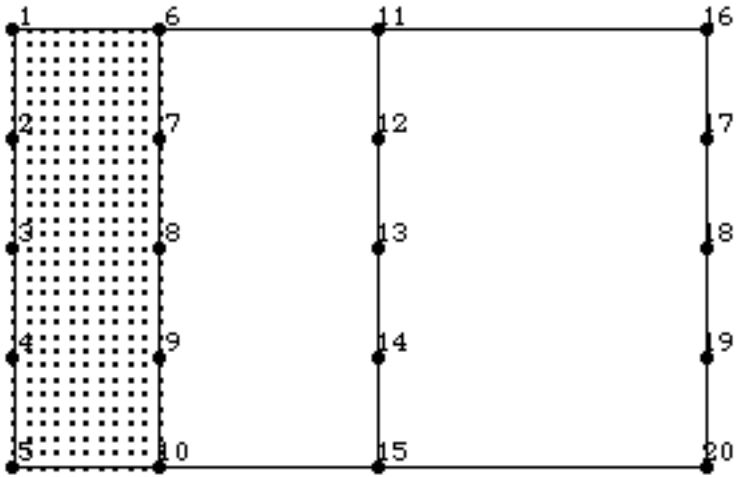
$$\alpha \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] = \left[\begin{array}{c|c} \hat{K}_{22} & \hat{K}_{23} \\ \hline \hat{K}_{32} & \hat{K}_{33} \end{array}\right]$$

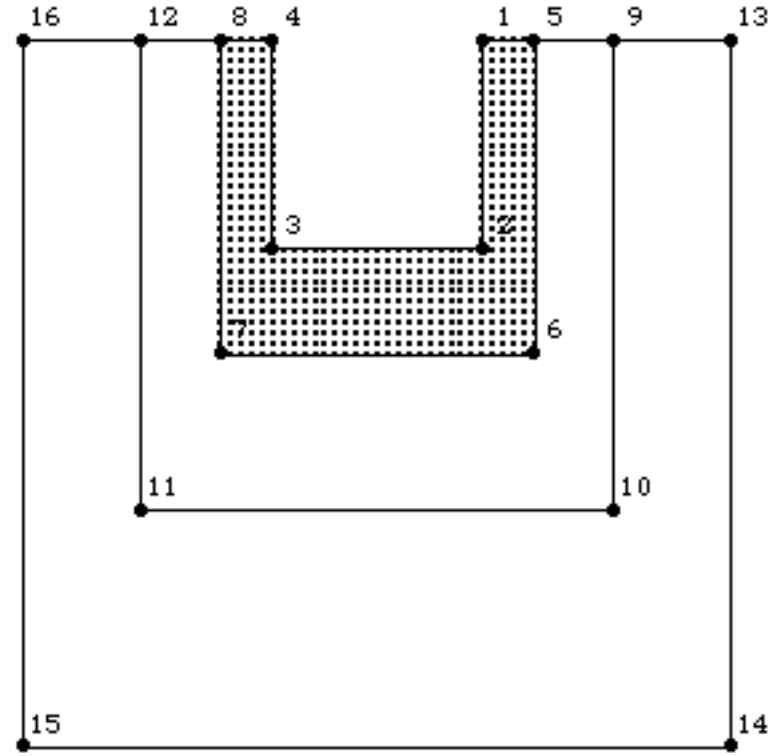


Finite domains:

$$\alpha \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] = \left[\begin{array}{c|c} \hat{K}_{22} & \hat{K}_{23} \\ \hline \hat{K}_{32} & \hat{K}_{33} \end{array}\right]$$



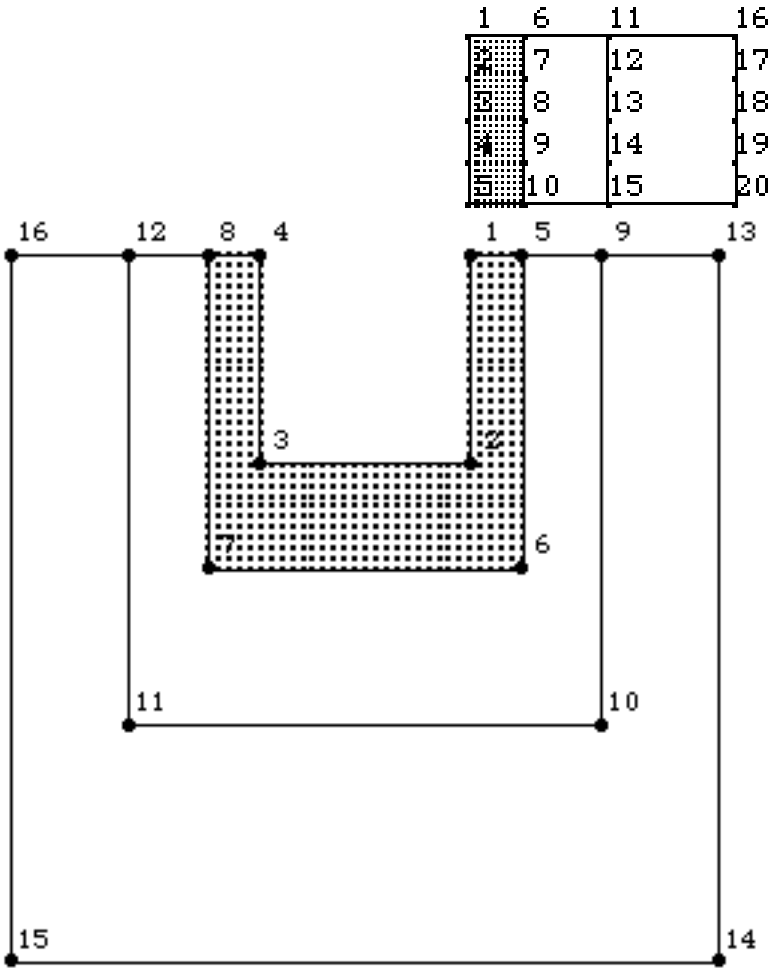




$$\left[\begin{array}{ccccc} K_{11} & K_{12} & \dots & 0 & 0 \\ K_{21} & K_{22}+\alpha K_{11} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha^{n-1}K_{22}+\alpha^{n}K_{11} & \alpha^{n}K_{12} \\ 0 & 0 & \dots & \alpha^{n}K_{21} & \alpha^{n}K_{22} \end{array}\right] \left\{\begin{array}{c} \{u_1\} \\ \{u_2\} \\ \vdots \\ \{u_{n-1}\} \\ \{u_n\} \end{array}\right\} = \left\{\begin{array}{c} \{f_1\} \\ \{0\} \\ \vdots \\ \{0\} \\ \{f_n\} \end{array}\right\}$$

Semi-finite domain:

$$\alpha \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] = \left[\begin{array}{c|c} \hat{K}_{22} & \hat{K}_{23} \\ \hline \hat{K}_{32} & \hat{K}_{33} \end{array}\right]$$



Define a transfer matrix:

$$\begin{aligned} \{\mathbf{u_2}\} &= [\mathbf{X_{21}}]\{\mathbf{u_1}\} \\ ([\mathbf{K_{11}}] + [\mathbf{K_{12}}][\mathbf{X_{21}}])\{\mathbf{u_1}\} &= \{\mathbf{f_1}\} = [\mathbf{K_1^*}]\{\mathbf{u_1}\} \\ ([\mathbf{K_{21}}] + [\mathbf{K_{22}}][\mathbf{X_{21}}])\{\mathbf{u_1}\} &= \{\mathbf{f_2}\} = -[\mathbf{K_2^*}][\mathbf{X_{21}}]\{\mathbf{u_1}\} \end{aligned}$$

Where $[\mathbf{K}^*]$ *is the stiffness of the unbounded domain.*

$$\text{IF}: \quad [\mathbf{K}^*] = [\mathbf{K_1^*}] \quad \text{and} \quad [\mathbf{Q_{21}}][\mathbf{K}^*] = [\mathbf{K_2^*}]$$

The transfer matrix and stiffness of the unbounded domain are defined by the stiffness of a single cell and the linear relationship between stiffnesses of neighboring cells.

$$[\mathbf{K_{12}}] + ([\mathbf{K_{22}}] + [\mathbf{Q_{21}}][\mathbf{K_{11}}])[\mathbf{X_{21}}] + [\mathbf{K_{21}}][\mathbf{X_{21}}][\mathbf{X_{21}}] = \mathbf{0}$$

Solving the quadratic matrix equation

$$[\mathbf{K_{12}}] + ([\mathbf{K_{22}}] + [\mathbf{Q_{21}}][\mathbf{K_{11}}])[\mathbf{X_{21}}] + [\mathbf{K_{21}}][\mathbf{X_{21}}][\mathbf{X_{21}}] = \mathbf{0}$$

Eigenvalue decomposition: $[\mathbf{X_{21}}] = \lambda_{\mathbf{i}}^{\mathbf{21}} \phi_{\mathbf{i}}^{\mathbf{21}}$

$$\begin{bmatrix} [0] & [I] \\ [Q_{21}].[K_{12}] & [K_{22}] + [Q_{21}][K_{11}] \end{bmatrix} \begin{Bmatrix} \phi_i^{21} \\ \lambda_i^{21}\phi_i^{21} \end{Bmatrix} = \lambda_{\mathbf{i}}^{\mathbf{21}} \begin{bmatrix} [I] & [0] \\ [0] & -[K_{21}] \end{bmatrix} \begin{Bmatrix} \phi_i^{21} \\ \lambda_i^{21}\phi_i^{21} \end{Bmatrix}$$

Provided the stiffnesses are symmetric:

$\lambda_i^{21} = 1$ rigid body mode

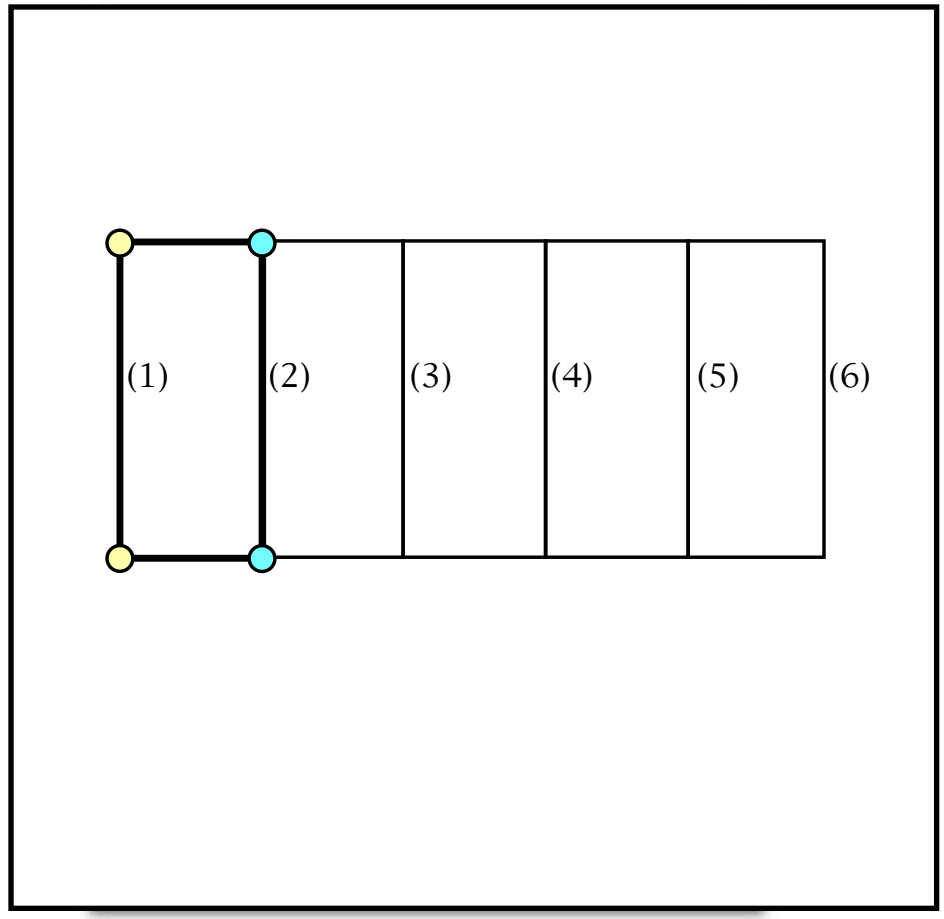
in general the eigenvalues are paired:

$|\lambda_i^{21}| < 1 \qquad |\lambda_i^{21}| > 1 \qquad \lambda_i^{21} = \dfrac{1}{\lambda_j^{21}}$

Translation: congruent elements

$$[\mathbf{K}^*] = [\mathbf{K}_\mathbf{1}^*] = [\mathbf{K}_\mathbf{2}^*]$$

- On a homogeneous domain the mechanical properties of each element is the same
- Equilibrium at the interior boundaries and the forces on the first and last element prescribe the domain behavior



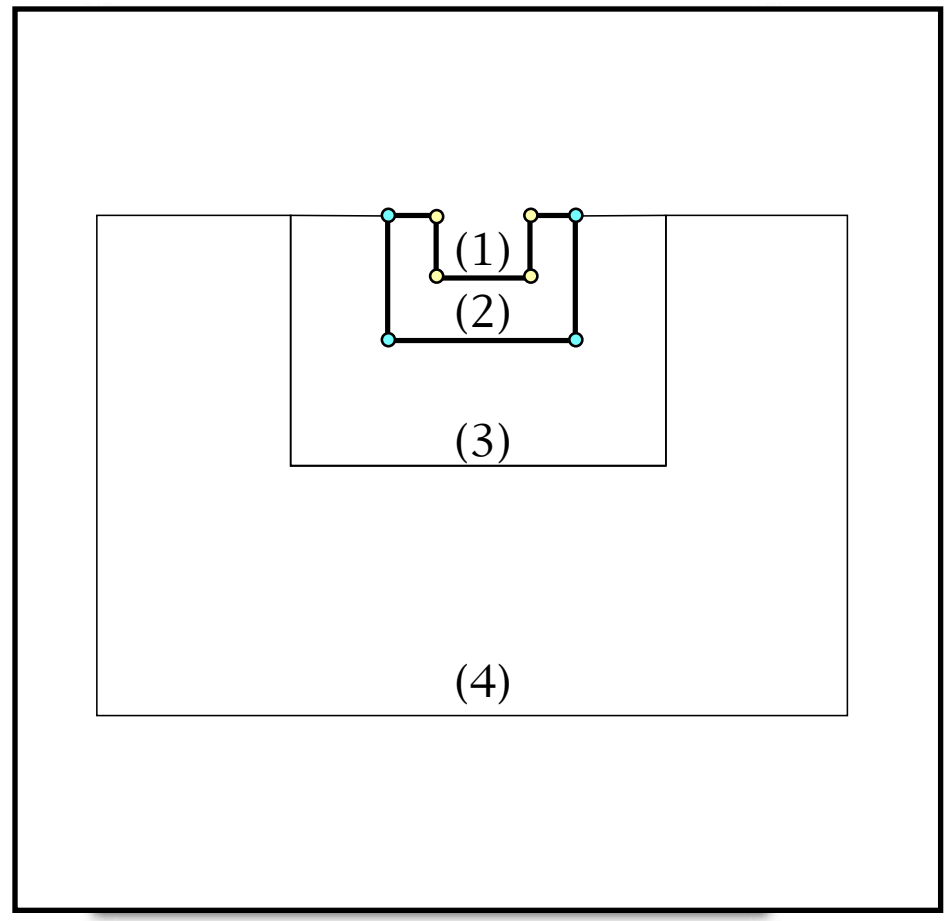
Scaling: similar elements

$$[\mathbf{K}^*] = [\mathbf{K}_1^*] = \frac{\mathbf{1}}{\xi}[\mathbf{K}_2^*]$$

- On a homogeneous domain the mechanical properties are proportional

$$< \mathbf{x}^*,\ \mathbf{y}^*,\ \mathbf{h} >= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{m} & \mathbf{n} & \xi \end{bmatrix} < \mathbf{x},\ \mathbf{y},\ \mathbf{1} >$$

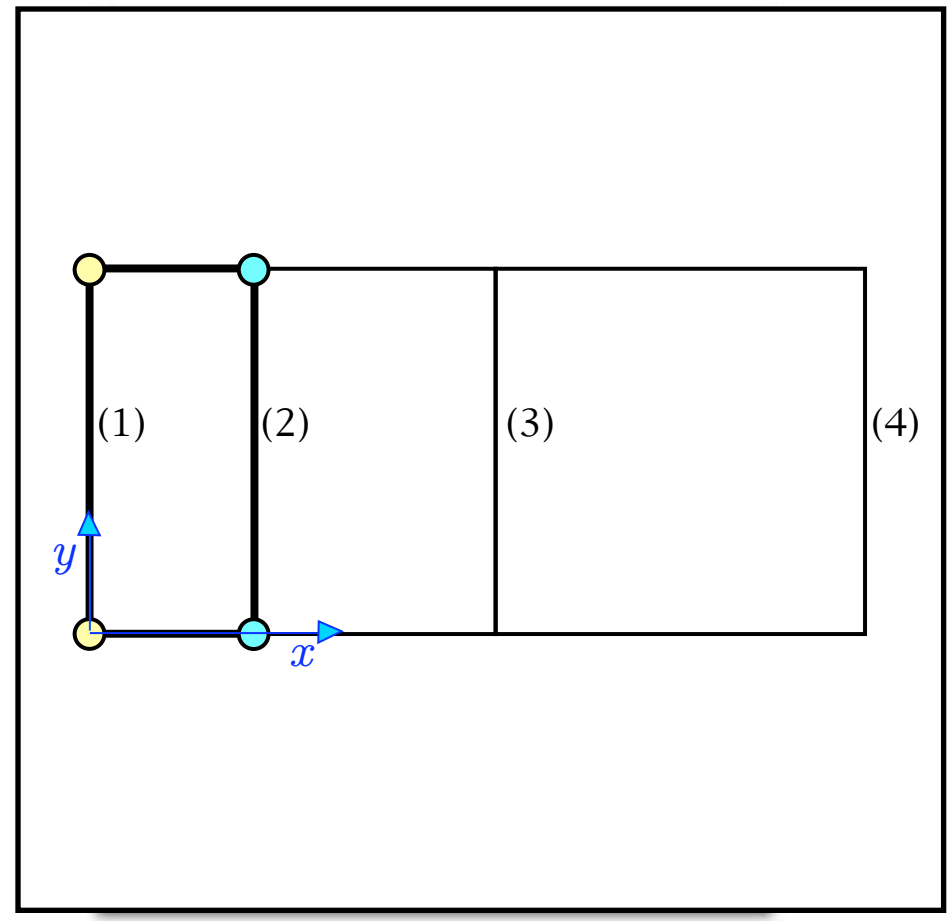
$$\mathbf{x}' = \frac{\mathbf{x}^*}{\mathbf{h}} = \frac{\mathbf{x}+\mathbf{m}}{\xi} \quad \text{and} \quad \mathbf{y}' = \frac{\mathbf{y}^*}{\mathbf{h}} = \frac{\mathbf{y}+\mathbf{m}}{\xi}$$



Affine transform: angles are congruent

$$[\mathbf{P}^*] + [\mathbf{K}^*] = [\mathbf{P}_1^*] + [\mathbf{K}_1^*] \quad \text{and} \quad \alpha[\mathbf{P}^*] + \beta[\mathbf{K}^*] = [\mathbf{P}_2^*] + [\mathbf{K}_2^*]$$

- On a homogeneous domain the principal components of the mechanical properties are proportional



$$< \mathbf{x}^*,\ \mathbf{y}^*,\ \mathbf{h} >= \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{0} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{0} \\ \mathbf{m} & \mathbf{n} & \xi \end{bmatrix} < \mathbf{x},\ \mathbf{y},\ \mathbf{1} >$$

$$\mathbf{x}' = \frac{\mathbf{x}^*}{\mathbf{h}} = \frac{\mathbf{a_{11}x} + \mathbf{a_{12}y} + \mathbf{n}}{\xi} \quad \text{and} \quad \mathbf{y}' = \frac{\mathbf{y}^*}{\mathbf{h}} = \frac{\mathbf{a_{12}x} + \mathbf{a_{22}y} + \mathbf{m}}{\xi}$$

... affine transformation continued

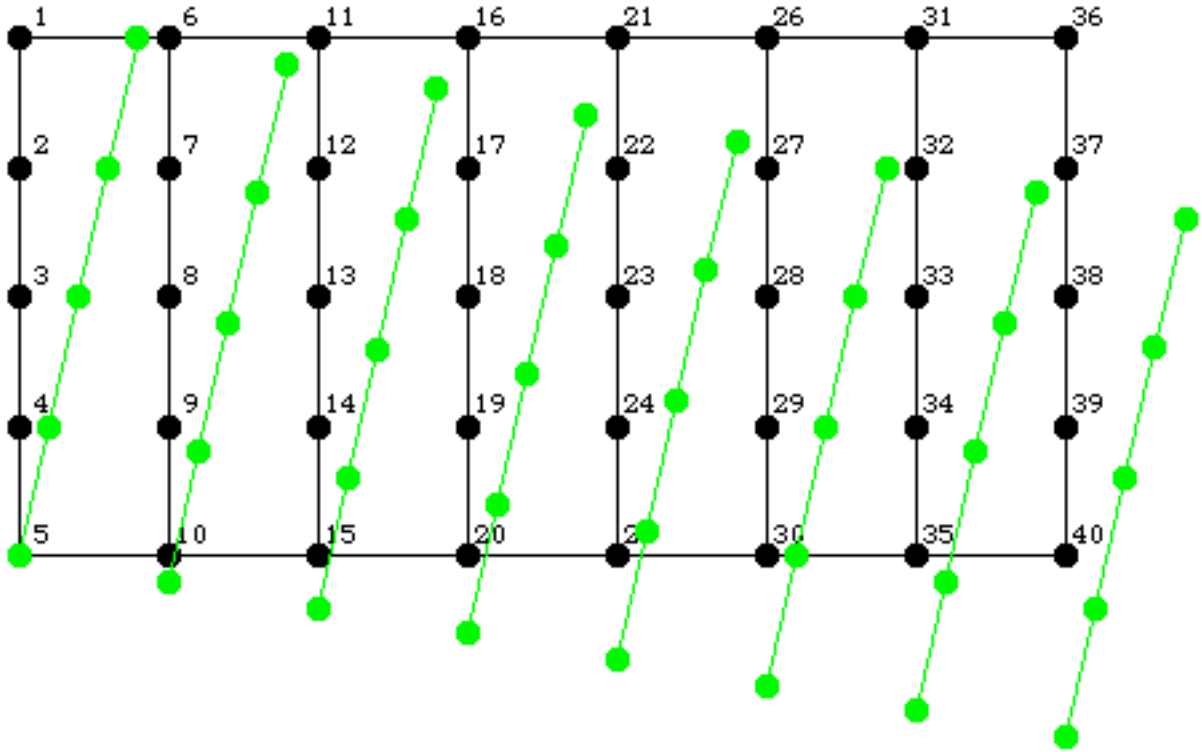
$$[\mathbf{K}^*] + [\mathbf{P}^*] = [\mathbf{K_1^*}] + [\mathbf{P_1^*}] \quad \text{and} \quad \alpha[\mathbf{K}^*] + \beta[\mathbf{P}^*] = [\mathbf{K_2^*}] + [\mathbf{P_2^*}]$$

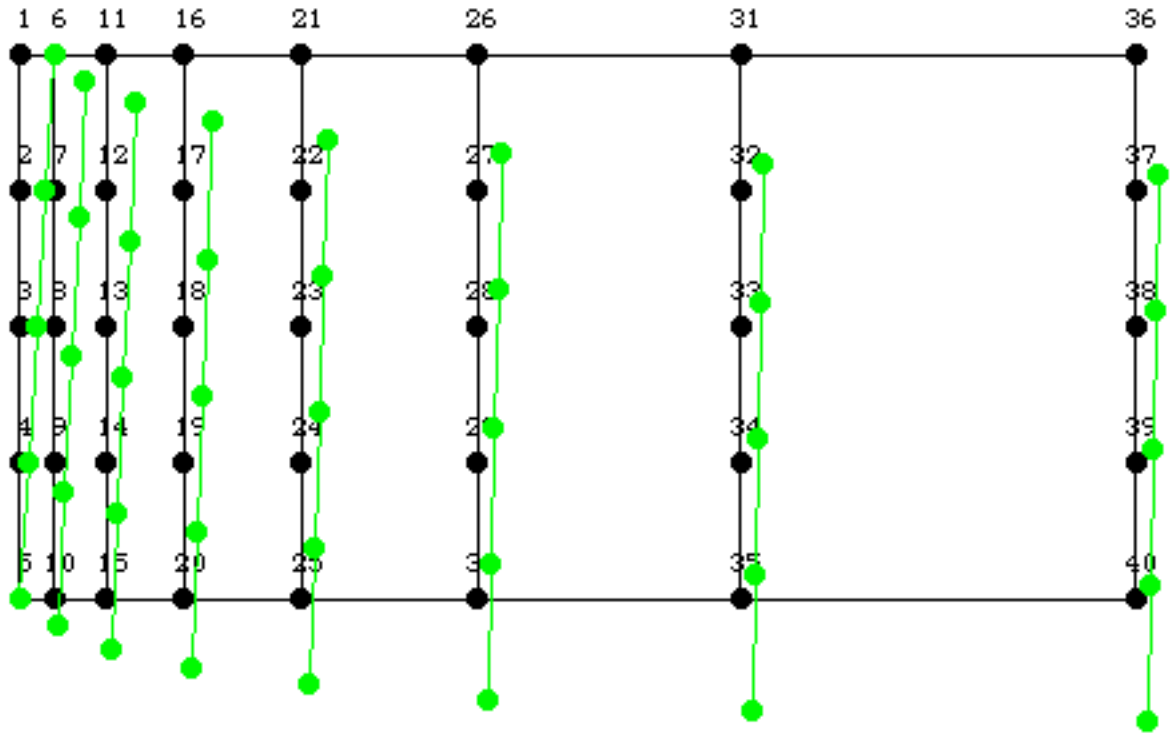
Solve the rigid body translation case with $\beta = 0$ and use the solution $[K^*]$ to solve $[K^*] + [P^*]$ for a different given β.

$$[\mathbf{D_{ij}}] = [\mathbf{P_{ij}}] + [\mathbf{K_{ij}}]$$

$$\begin{bmatrix} [0] & [I] \\ \beta[D_{12}] & [D_{22}] + \beta[D_{11}] + (\alpha - \beta)[K^*] \end{bmatrix} \begin{Bmatrix} \phi_i^{21} \\ \lambda_i^{21}\phi_i^{21} \end{Bmatrix} = \lambda_{\mathbf{i}}^{\mathbf{21}} \begin{bmatrix} [I] & [0] \\ [0] & [D_{21}] \end{bmatrix} \begin{Bmatrix} \phi_i^{21} \\ \lambda_i^{21}\phi_i^{21} \end{Bmatrix}$$

Solution is valid if $[K^*]$ is independent of β and $[\mathbf{K_{12}}] = [\mathbf{K_{21}}]^{\mathbf{T}}$ is not singular.

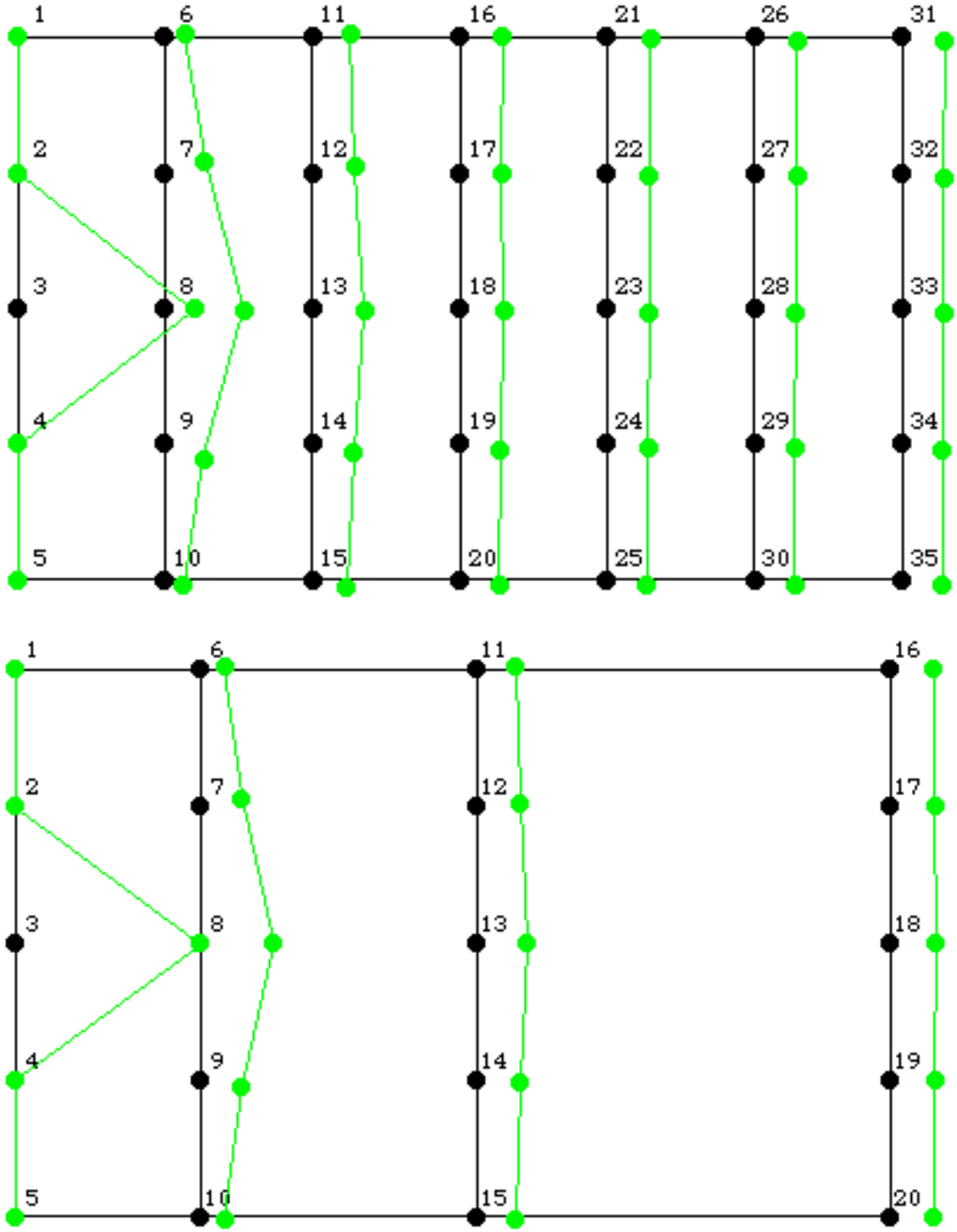




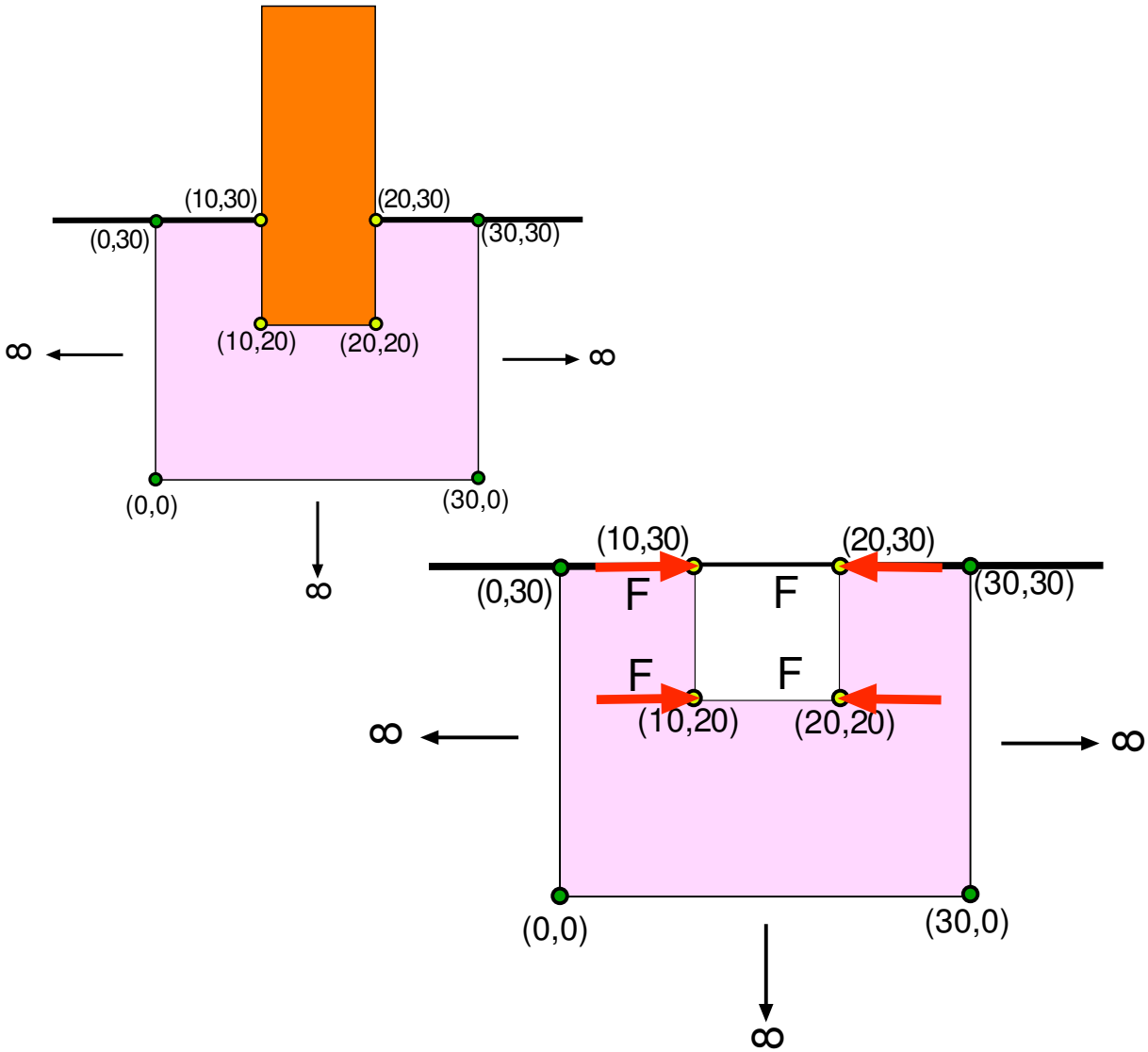
Rigid body modes in two dimensions:

If the shapes are too similar the rigid body rotation mode is not damped. Rigid body translation modes are similarly undamped if the material is homogeneous.

Some static results: local disturbance



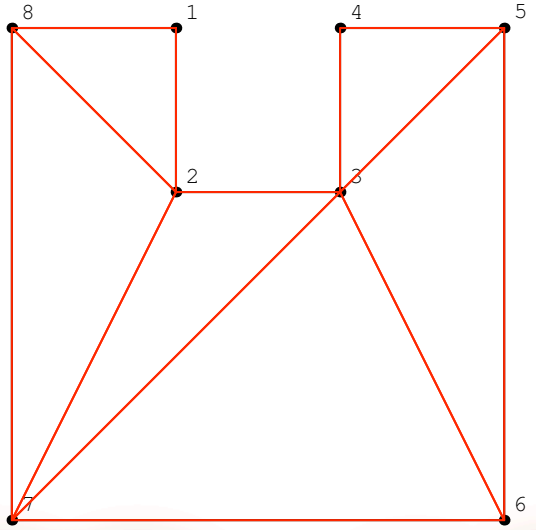
Application: Idealized wind turbine

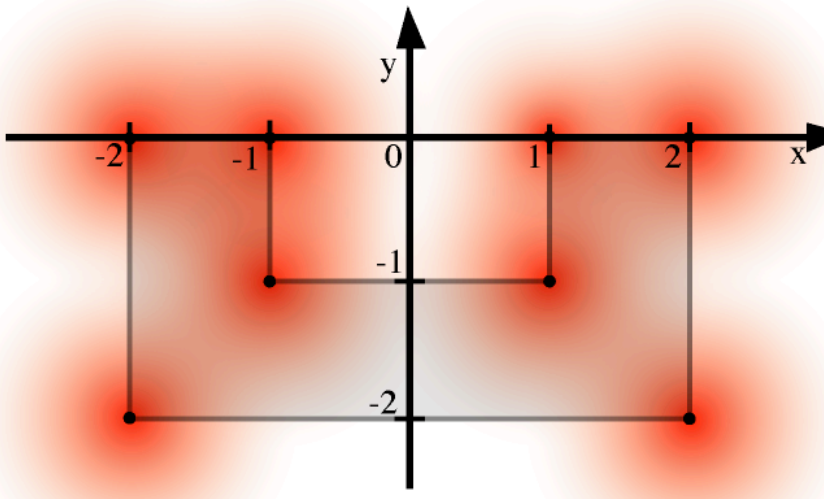


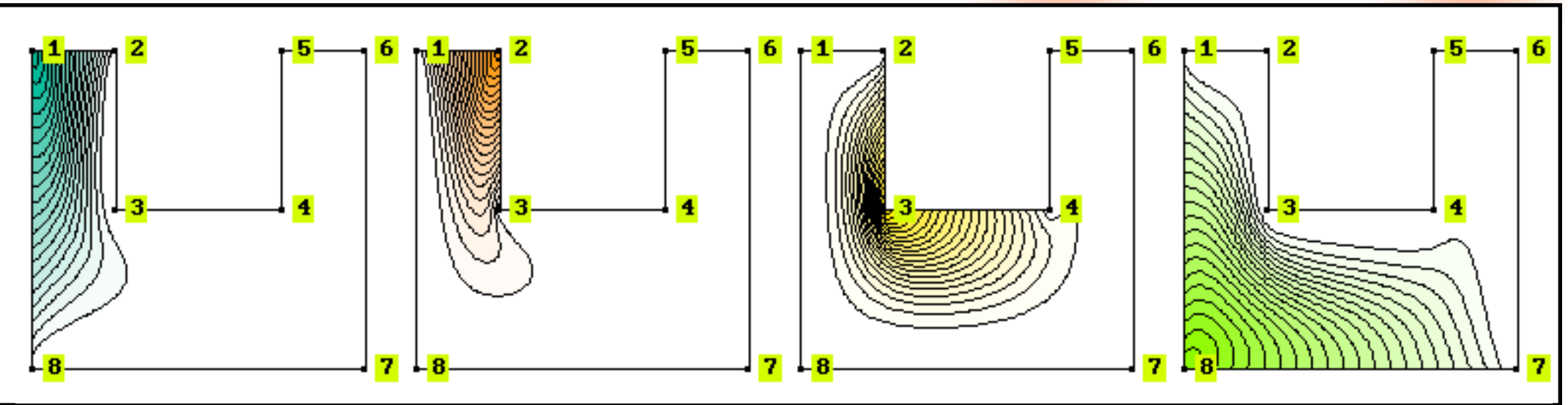
- The wind turbine structure is modeled with conventional finite elements
- The footing is necessarily concave -- U shaped --
- Ground modeled using scaled elements

Modeling the concave cell:

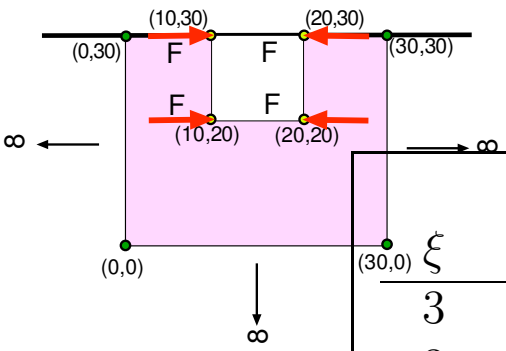
- Triangulation
- Non-conformal methods (BEM, Meshless)
- Reduce cell thickness to a line (SBFEM)
- Shape functions for a concave element (Mean value coordinates, MCM)



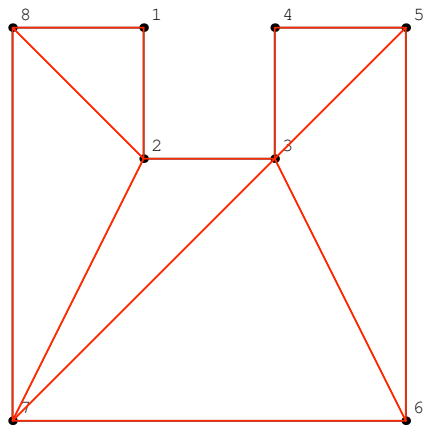


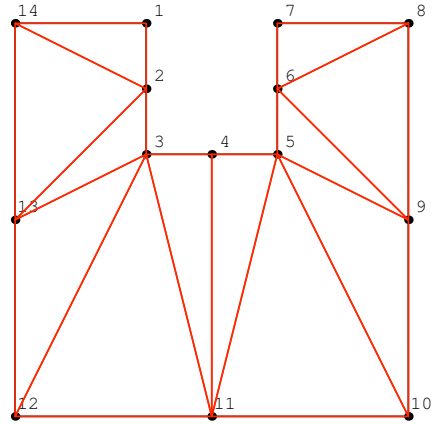


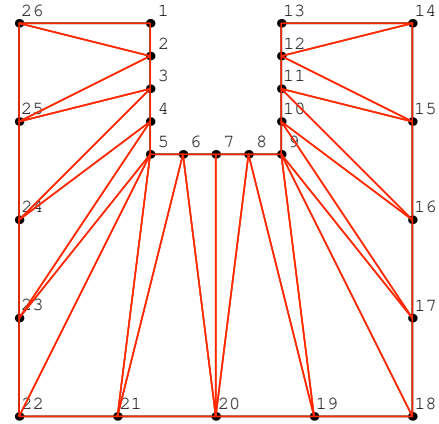
Mesh or no mesh



ξ	side nodes	REP. CELL min. eig.	REP. CELL max. eig.	UNBOUNDED CELL min. eig.	UNBOUNDED CELL max. eig.	max disp $[m]$.
3	0	13.74	704.3	36.40	539.0	0.1284
3	1	6.591	1001	27.57	877.8	0.1302
3	3	6.055	2439	18.53	2267	0.1038
1.5	0	4.301	856.4	39.02	357.0	0.09602
1.5	1	0.5696	648.3	16.30	291.0	0.1819
1.5	3	0.3308	744.3	8.290	506.3	0.2232
1.1	0	0.920	3175	45.01	388.4	0.0813
1.1	1	0.1530	2622	18.97	349.6	0.1465
1.1	3	0.02484	1379	7.827	292.5	0.2200



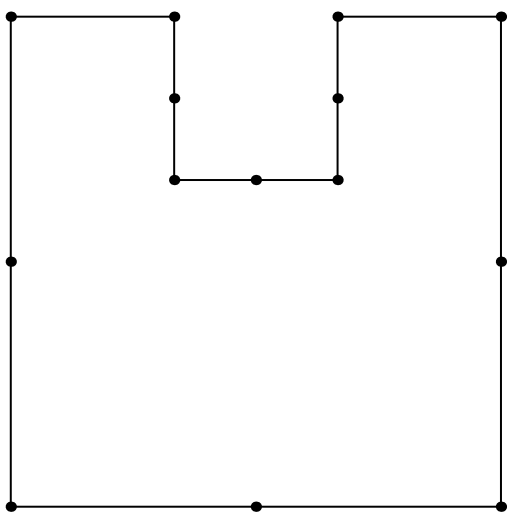


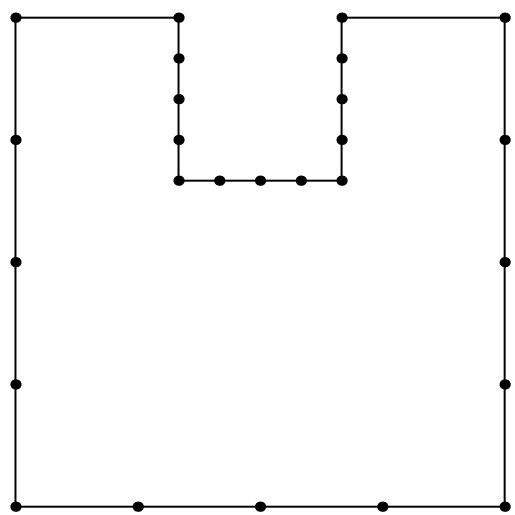


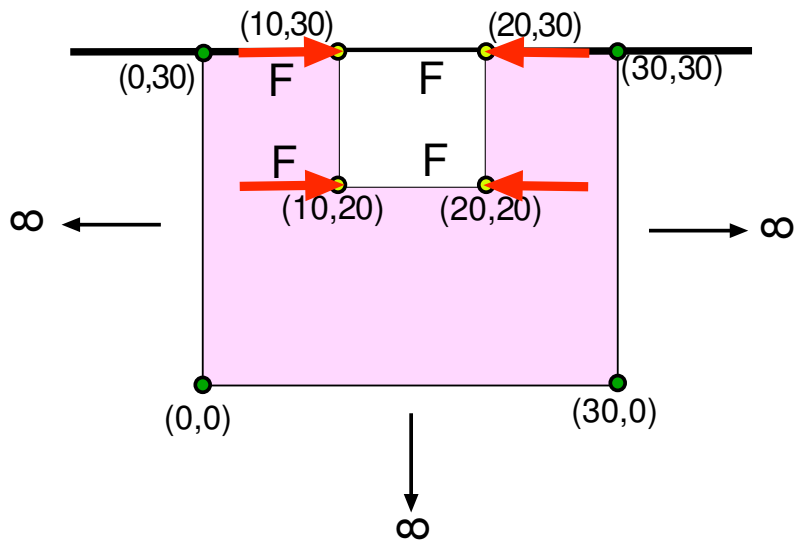
Numerical challenge:

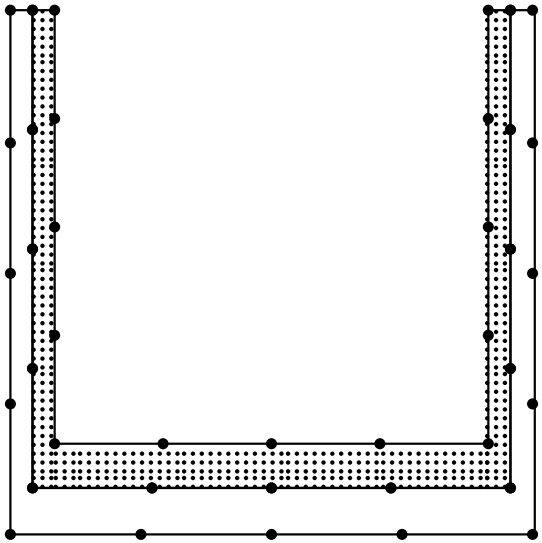
ξ	step	h	REP. CELL min. eig.	REP. CELL max. eig.	UNBOUNDED CELL min. eig.	UNBOUNDED CELL max. eig.	max. disp.
1.5	0.5	0.2	4.776	870.1	34.44	191.9	0.1049
1.5	0.1	0.04	3.658	719.8	28.72	148.65	0.1237
1.1	0.1	0.2	0.5398	3600	32.34	219.3	0.1079

ξ	step	h	REP. CELL min. eig.	REP. CELL max. eig.	UNBOUNDED CELL min. eig.	UNBOUNDED CELL max. eig.	max. disp.
Three side nodes.							
1.5	0.5	0.2	1.659	384.8	11.49	246.1	0.1470
1.1	0.2	0.4	0.0702	1577	8.592	252.1	0.1928
Seven side nodes.							
1.1	0.1	0.2	0.0279	827.1	4.344	183.2	0.2121

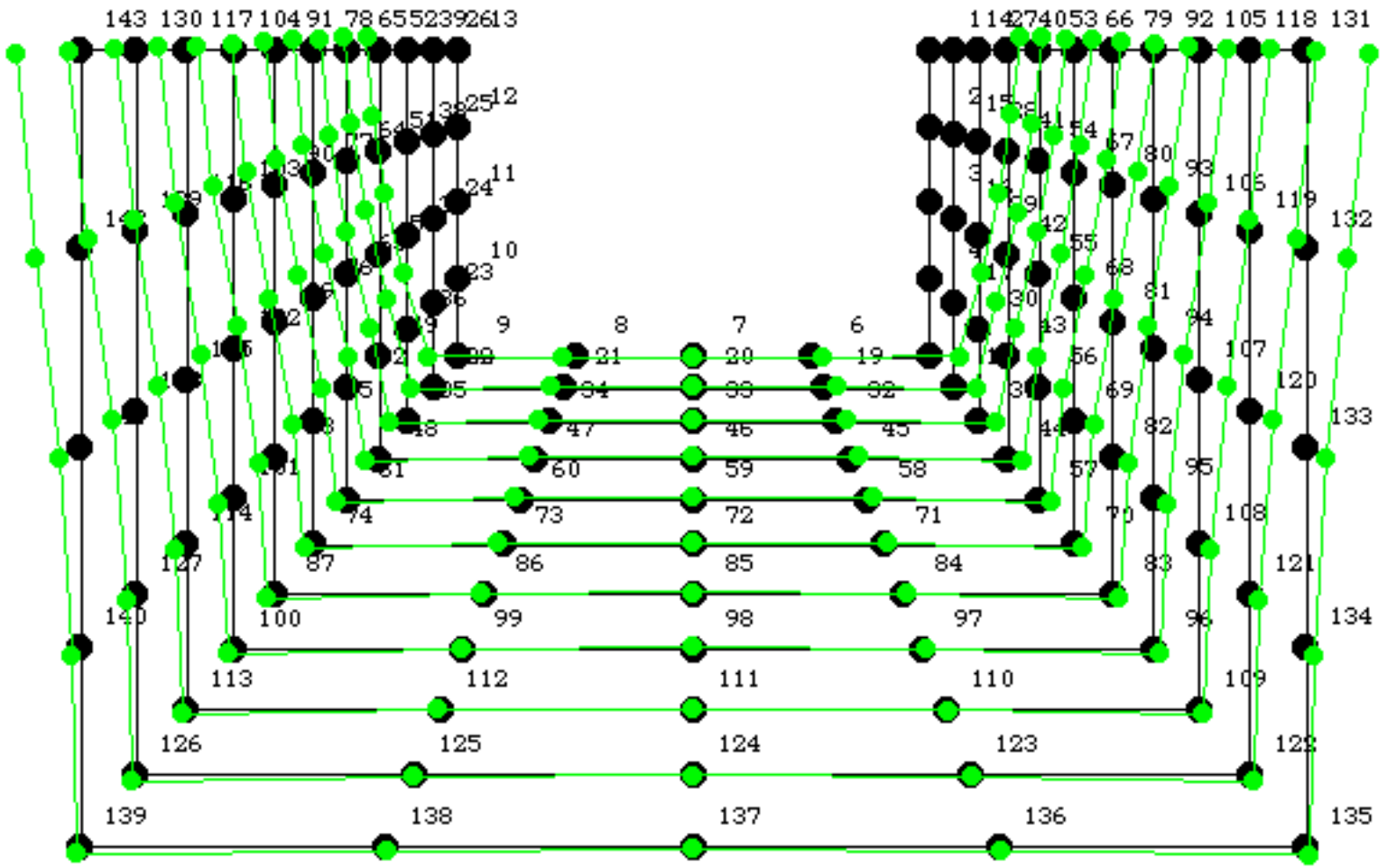






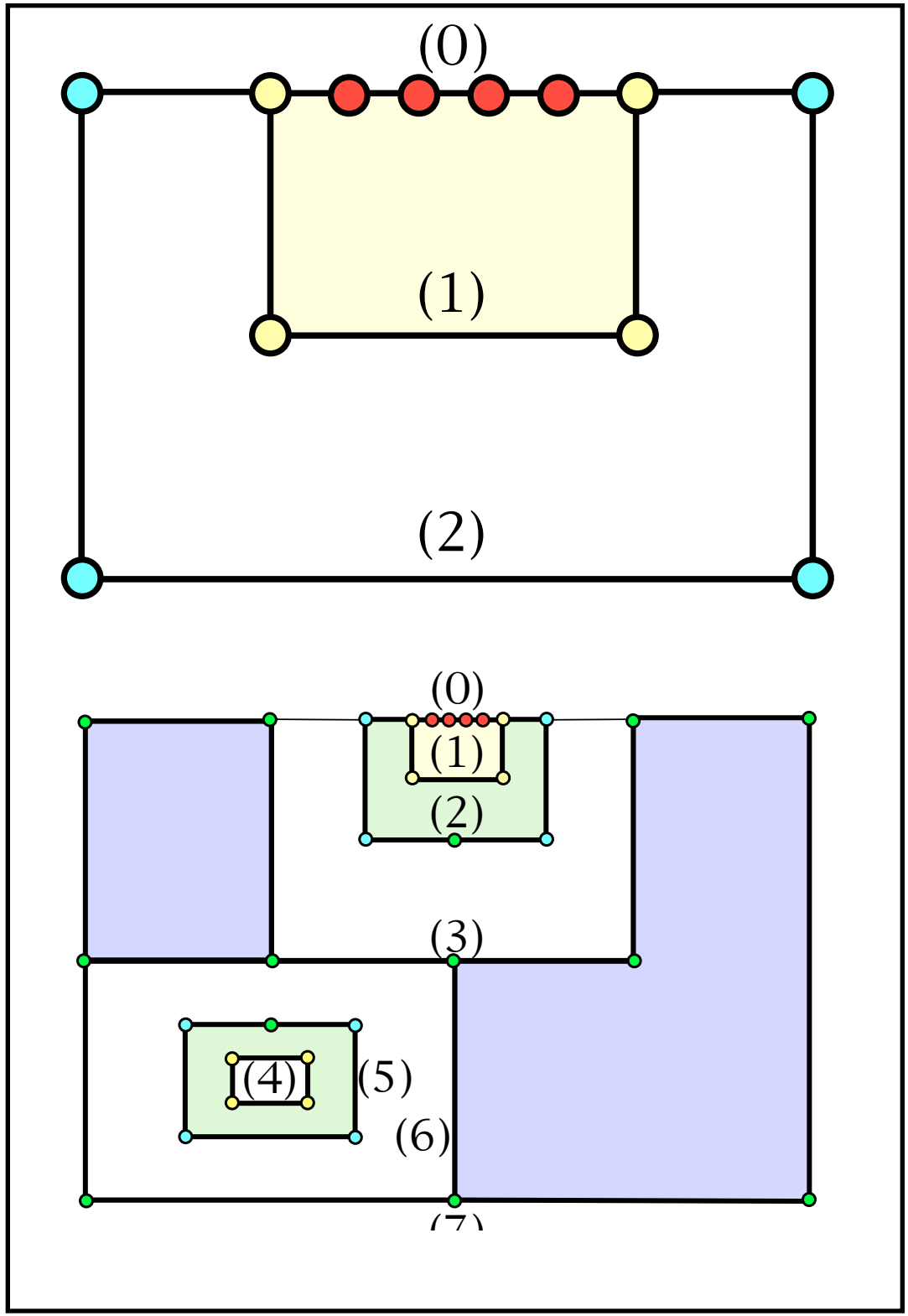


The representative cell prescribes the behavior of the domain.



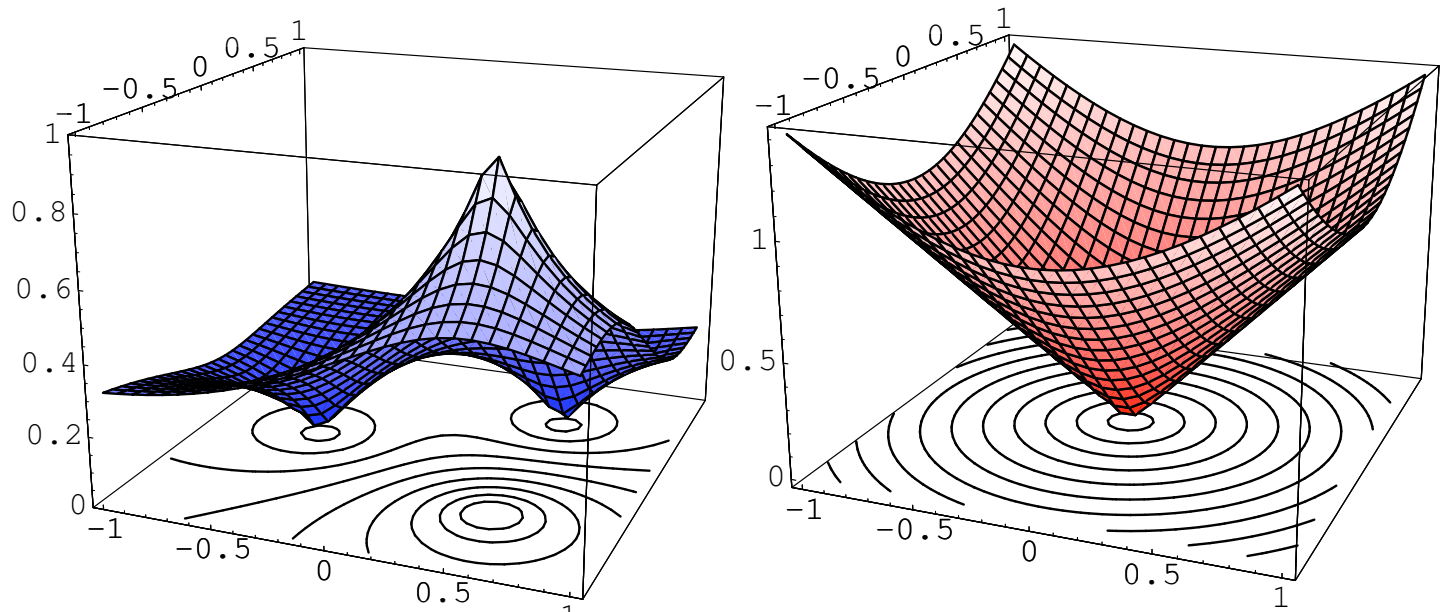
Coupled domains

- Both finite and infinite domain are modeled using finite elements
- Choose the shape functions and such that the meshes are conformal

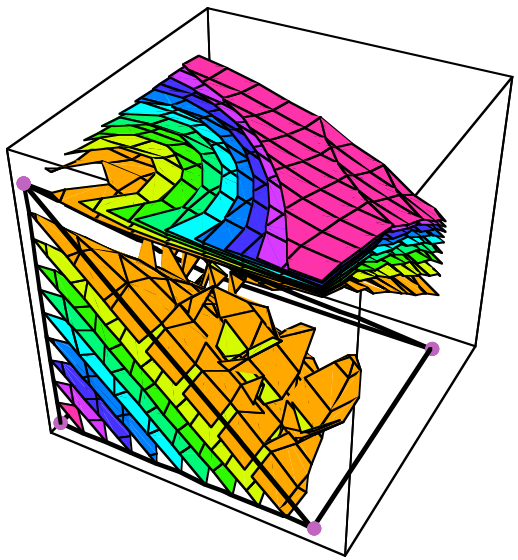


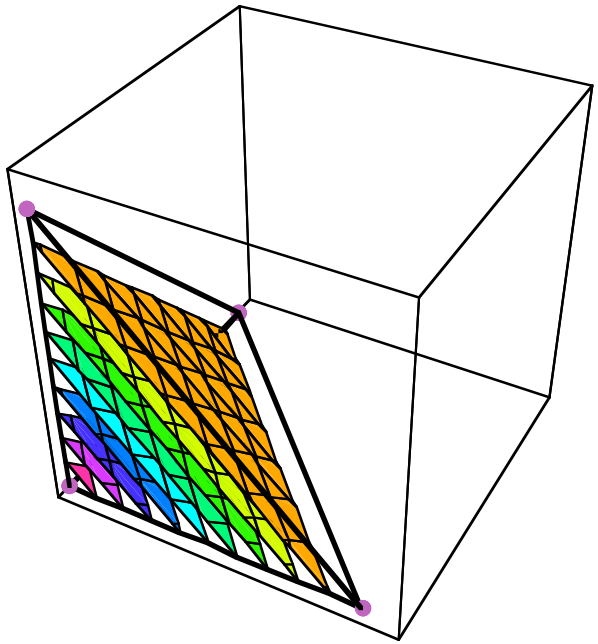
To do:

- Compare and formalize error estimates using rules for graded meshes
- Implement dynamic, frequency domain, model
- Apply to unusual material models (i.e. not elastic)
- 3-Dimensional domains

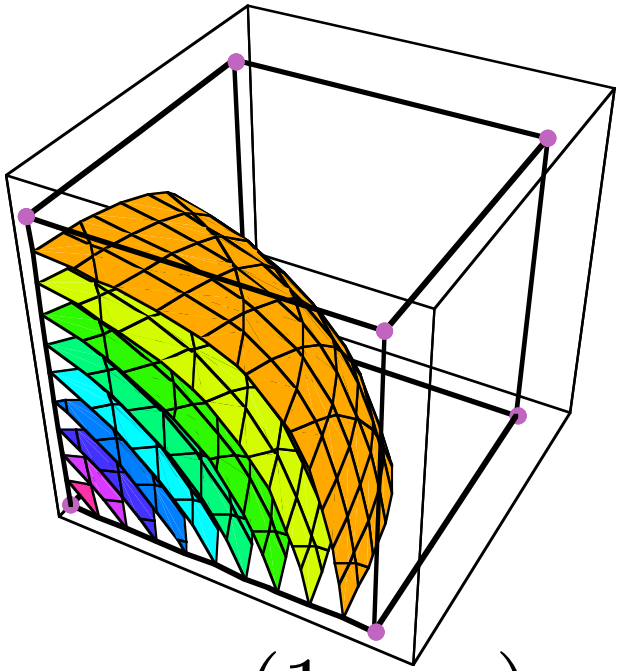


3D !



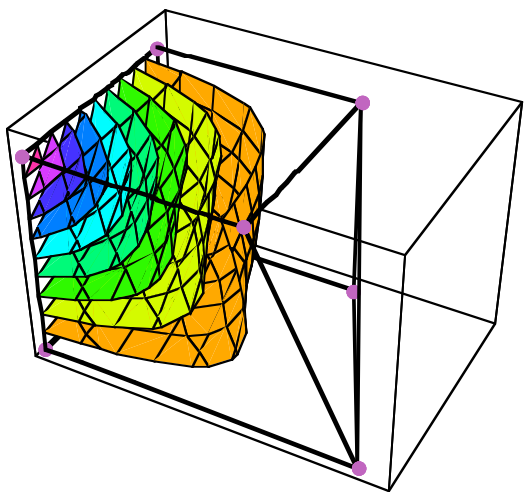


$$1 - x - y - z$$



$$x\,y\,(1 - z)$$

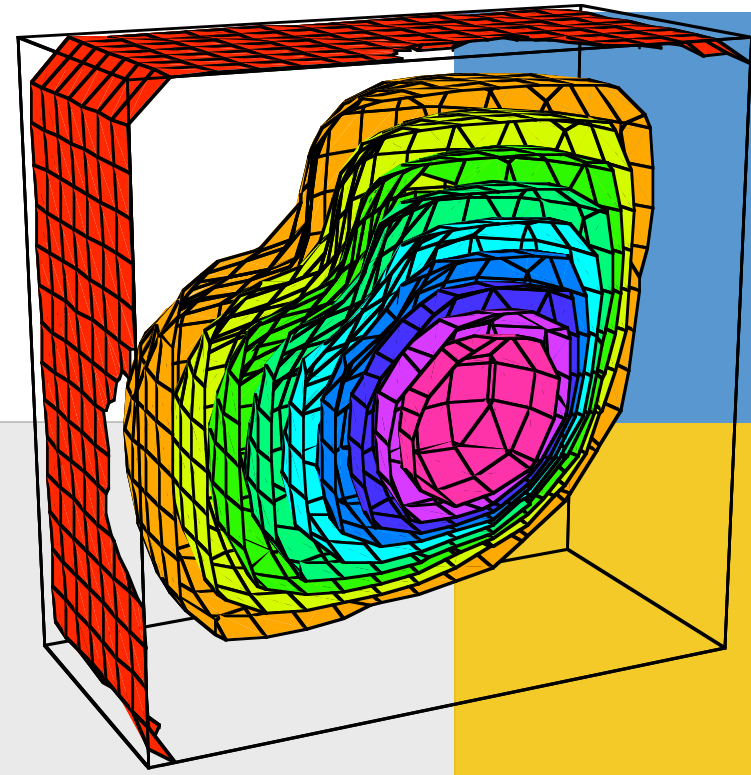
$$((x+z-1)\,(y+z-1)\,(2\,y^2\,x^2 - 2\,y\,x^2 - z\,x^2 - 2\,y^2\,x - z^2\,x + 2\,y\,x + z\,x - y\,z^2 - y^2\,z + y\,z)) \,/$$
$$(x\,z^3 + y\,z^3 + x^2\,z^2 + y^2\,z^2 - 2\,x\,z^2 - 2\,y\,z^2 - x^2\,z + x\,y^2\,z - y^2\,z + x\,z + x^2\,y\,z - 2\,x\,y\,z + y\,z + 2\,x^2\,y^2 - 2\,x\,y^2 - 2\,x^2\,y + 2\,x\,y)$$

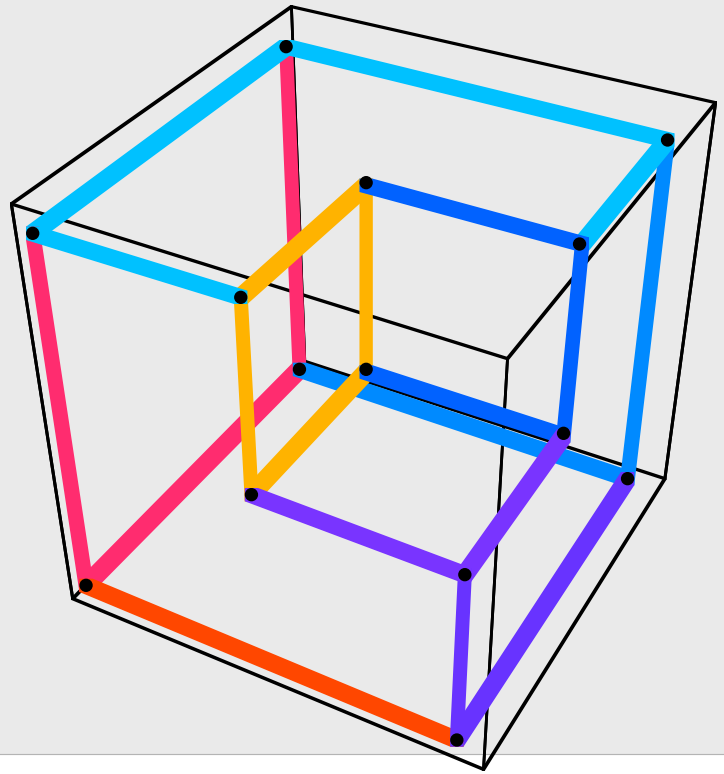


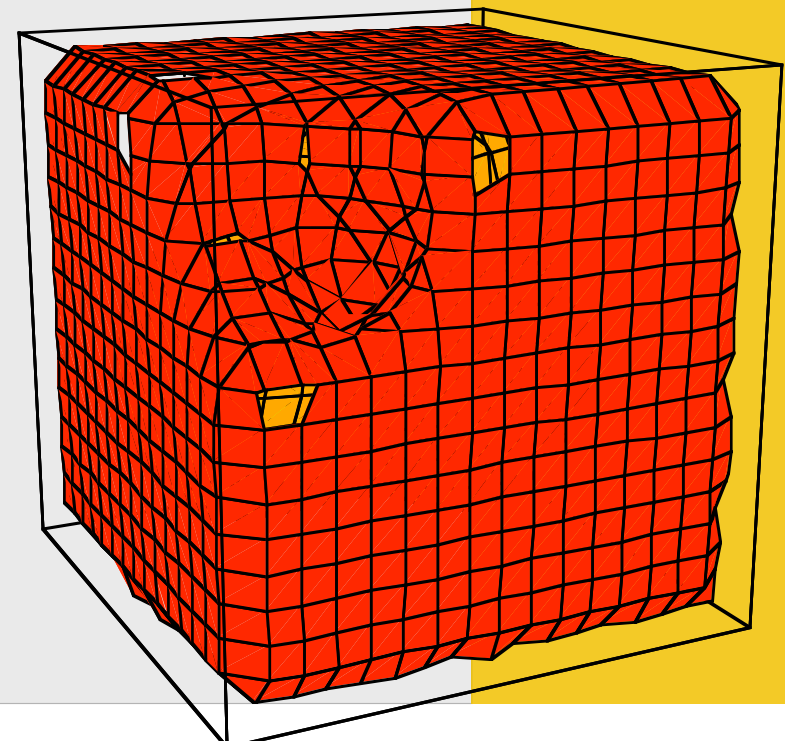
$$\ldots y + 7.5\,x\,y^2 - 3.75\,x^2\,y^2 - 1.875\,x\,y^3 - 1.6875\,z - 5.625\,x\,z + 4.5\,x^2\,z +$$
$$\ldots z + 2.8125\,z^2 + 6.75\,x\,z^2 - 4.5\,x^2\,z^2 - 3.75\,y\,z^2 - 4.5\,x\,y\,z^2 + 1.5\,x^2\,y\,z^2 + 0.9375\,y^2\,z^2 + 0.75\,x\,y^2\,z^2]]\,/$$
$$\ldots 531\,x^2\,y^2 - 16.875\,x^3\,y^2 + 22.1484\,x\,y^3 - 21.0938\,x^2\,y^3 + 4.21875\,x^3\,y^3 - 3.16406\,x\,y^4 + 2.10938\,x^2\,y^4 + 28.4766\ldots$$
$$\ldots 25\,x^2\,y\,z - 13.5\,x^3\,y\,z - 37.9688\,y^2\,z + 65.1797\,x\,y^2\,z - 41.3438\,x^2\,y^2\,z + 9.84375\,x^3\,y^2\,z + 17.7188\,y^3\,z -$$
$$\ldots z + 3.375\,x\,y^4\,z - 1.125\,x^2\,y^4\,z - 47.4609\,x\,z^2 + 56.9531\,x^2\,z^2 - 16.875\,x^3\,z^2 - 37.9688\,y\,z^2 + 65.1797\,x\,y\,z^2 -$$
$$\ldots 5\,x\,y^2\,z^2 + 12.0937\,x^2\,y^2\,z^2 - 2.25\,x^3\,y^2\,z^2 - 29.5313\,y^3\,z^2 + 15.5391\,x\,y^3\,z^2 - 1.96875\,x^2\,y^3\,z^2 + 0.28125\,x^3\,y^3\,z\ldots$$
$$\ldots 484\,x\,z^3 - 21.0938\,x^2\,z^3 + 4.21875\,x^3\,z^3 + 17.7188\,y\,z^3 - 26.1562\,x\,y\,z^3 + 12.9375\,x^2\,y\,z^3 - 2.25\,x^3\,y\,z^3 -$$
$$\ldots 25\,x^3\,y^2\,z^3 + 13.7813\,y^3\,z^3 - 3.9375\,x\,y^3\,z^3 - 4.44089 \times 10^{-16}\,x^2\,y^3\,z^3 - 1.96875\,y^4\,z^3 + 0.28125\,x\,y^4\,z^3 -$$
$$\ldots 1.125\,x^2\,y\,z^4 + 4.21875\,y^2\,z^4 - 1.61719\,x\,y^2\,z^4 + 0.140625\,x^2\,y^2\,z^4 - 1.96875\,y^3\,z^4 + 0.28125\,x\,y^3\,z^4 + 0.28125\,x\ldots$$

$$\hat{b}_i(\mathbf{x}) = \frac{b_i(\mathbf{x})}{\prod r_{jk}(\mathbf{x})} = \frac{a_i}{\prod_{j,k \in ind(i)} r_{jk}(\mathbf{x})}$$

$$\hat{b}_j^{mn}(\mathbf{x}) = \frac{a_j}{r_{ij}^m(\mathbf{x}) r_{jk}^n(\mathbf{x})}$$







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Thank You.