Test functions for elliptic problems satisfying linear essential edge conditions on both convex and concave polygons

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In **1877** Lord Rayleigh developed the approximation method for problems of vibration.



In **1908** Walter Ritz independently presented the general approximation method.



$$\mathcal{L}[\vec{u}(x,y)] \approx [\mathsf{A}]^{n\times n}\{\vec{u}_n\} = \{\vec{f}_n\}$$

Construct an approximate solution to any **elliptic partial differential equation**

construct variational form

↓

assume a solution

↓

minimize

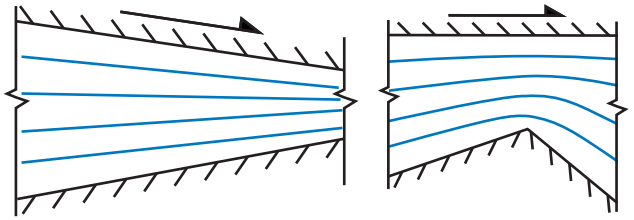
The **test function** is a particular assumed solution which satisfies the following properties:

- Smoothness
- Essential boundary conditions
- Linear independence

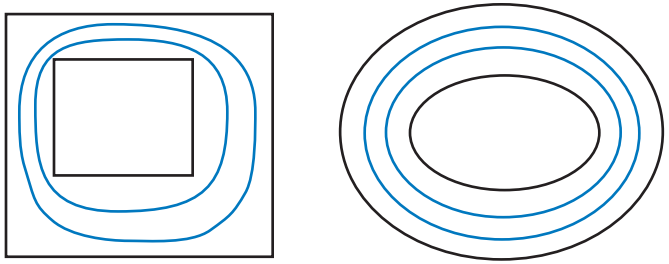
Solution to a homogenous elliptic problem

Maximum and minimum values *must* occur on the boundary.
Functions *must* be smooth in the interior.

Examples:



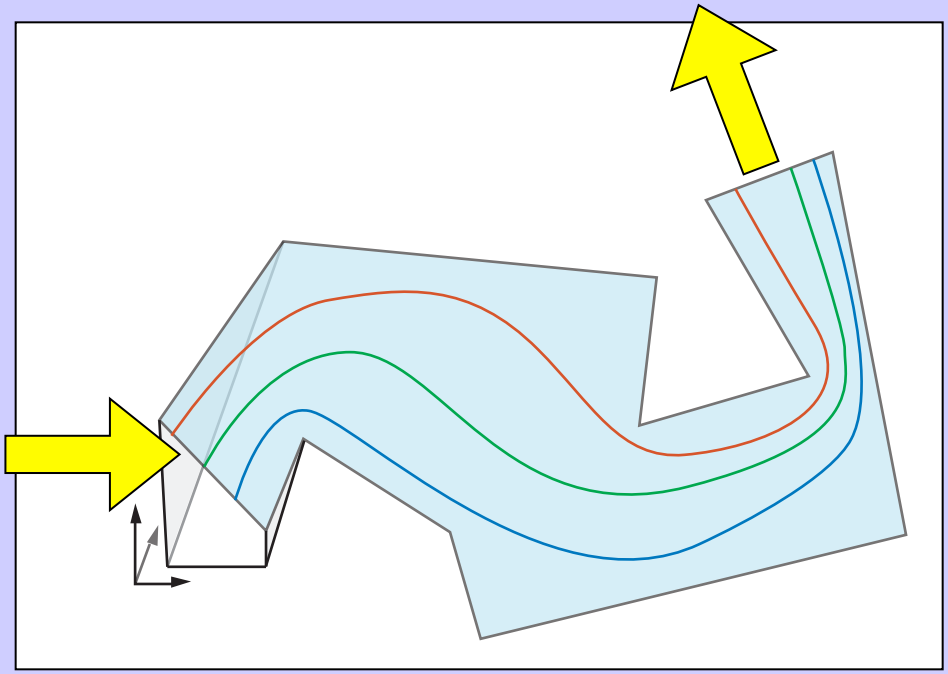
• Irrotational fluid flow



• Torsion stress function/ Membrane

• Temperature distribution ...

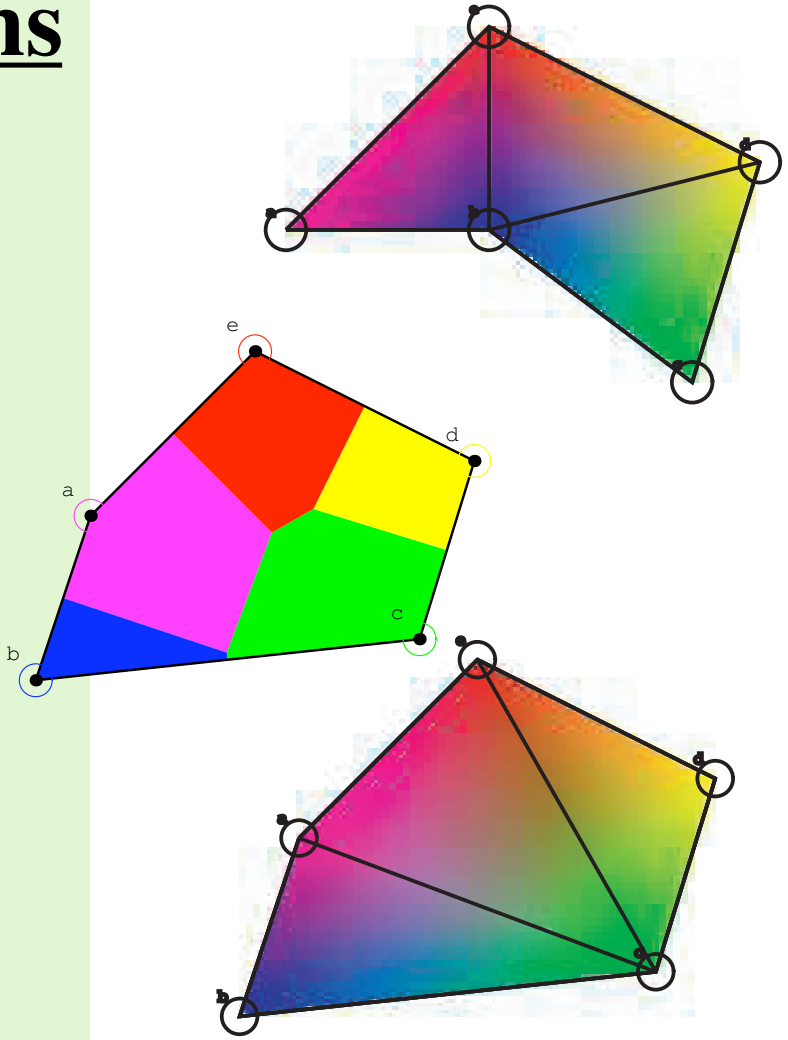
$$\nabla^2 u(x, y) = 0$$



Racetrack analogy
an elliptic race track would:
Never let a runner run uphill
Lanes only bend smoothly
Lanes do not cross

Available 2-D test functions

- Some simple two dimensional domains
  - Courant's triangle
  - Taig's quadrilateral
- Wachspress convex element (1975)

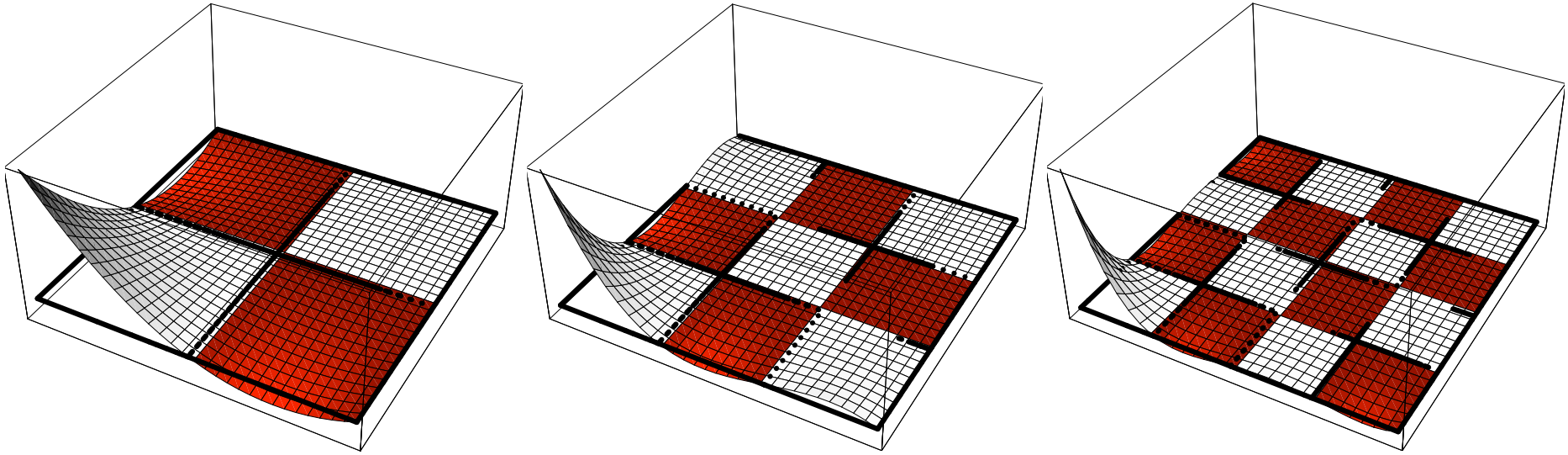


Alternative approaches

- *Finite Element Method*
  - sacrifice smoothness
- *Boundary Element Method*
  - sacrifice boundary conditions
  - solve the field equations

The test function conditions are satisfied only in an average sense.

Counter Example:
Higher order Lagrange elements



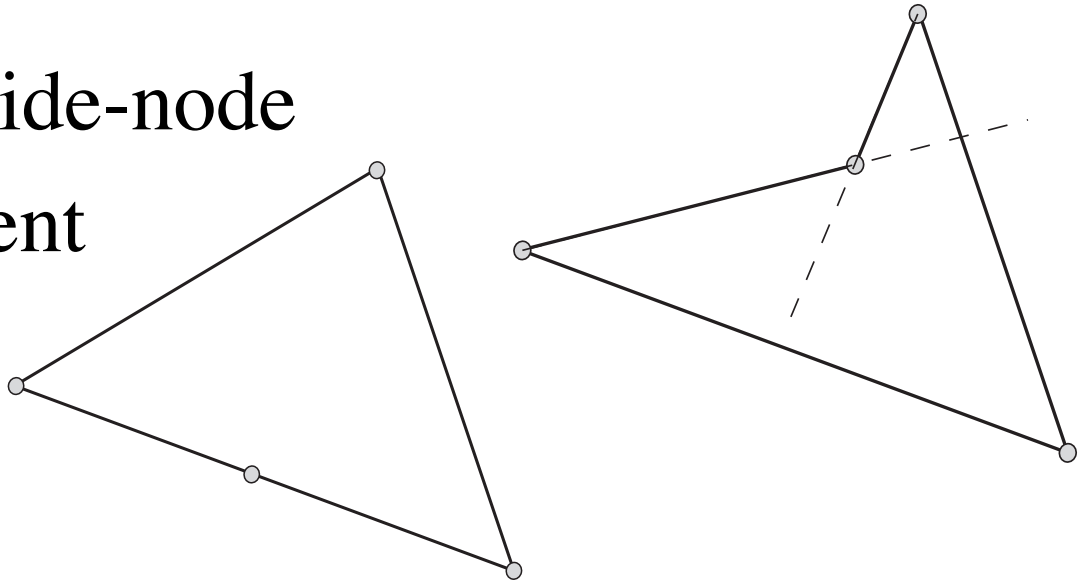
Unrealistic representations: Negative Kelvin

Inability to reproduce constant and linear fields:
rigid body motion produces strain energy

Non-uniform convergence: instability

Unavailable test function formulations

- Linear edged side-node
- Concave element
- Inclusions

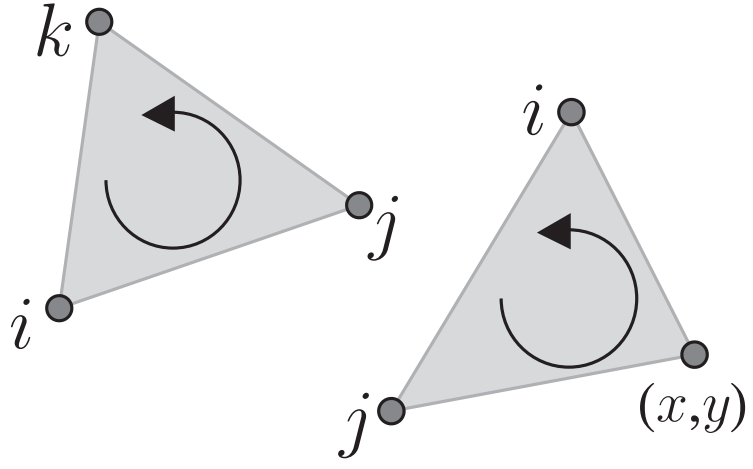


Polynomial and rational polynomial formulations are insufficient

$$\frac{a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy}{b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 y^2 + a_5 xy}$$

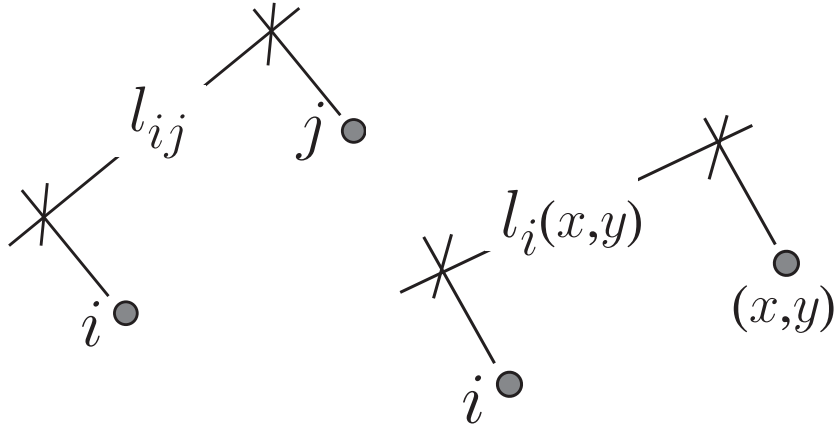
Other methods, such as conformal mapping apply only to specific shapes and result in transcendental functions, not in algebraic functions.

Area



$$\triangle(i,j,k) = \frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}$$

Length



$$l_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$